

THE DIRICHLET PROBLEM FOR CURVATURE EQUATIONS IN RIEMANNIAN MANIFOLDS

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ABSTRACT. We prove the existence of classical solutions to the Dirichlet problem for a class of fully nonlinear elliptic equations of curvature type on Riemannian manifolds. We also derive new second derivative boundary estimates which allows us to extend some of the existence theorems of Caffarelli, Nirenberg and Spruck [4] and Ivochkina, Trudinger and Lin [18], [19], [25] to more general curvature functions under mild conditions on the geometry of the domain.

1. INTRODUCTION

The aim of this article is to study the classical Dirichlet problem for equations of prescribed curvature of the form

$$(1.1) \quad F[u] = f(\kappa[u]) = \Psi(x, u)$$

defined on a smooth Riemannian manifold (M^n, σ) , $n \geq 2$, where $\kappa[u]$ is the vector in \mathbb{R}^n whose components $\kappa_1, \dots, \kappa_n$ are the principal curvatures of the graph $\Sigma = \{(x, u(x)), x \in \Omega\} \subset M \times \mathbb{R}$ of a function u defined on a bounded domain $\Omega \subset M$, Ψ is a prescribed positive function on $\bar{\Omega} \times \mathbb{R}$ and f is a general curvature function in a sense that it will be made precise later. The main examples of general curvature functions are given by the k -th root of the higher order mean curvatures

$$(1.2) \quad S_k(\kappa) = \sum_{i_1 < \dots < i_k} \kappa_{i_1} \cdots \kappa_{i_k}$$

and the $(k-l)$ -th root of their quotients $S_{k,l} = S_k/S_l$, $0 \leq l < k \leq n$. The mean, scalar, Gauss and harmonic curvatures correspond to the special cases $k = 1, 2, n$ in (1.2) and $k = n, l = n-1$ for the quotients, respectively.

The classical Dirichlet problem associated to equation (1.1) has been extensively studied (see for instance [4], [11], [13], [15], [18], [19], [20], [29], [31] and [32]). For domains in the Euclidean space, the first breakthroughs about the solvability of the Dirichlet problem

$$(1.3) \quad \begin{aligned} F[u] &= f(\kappa[u]) = \Psi && \text{in } \Omega \\ u &= \varphi && \text{on } \partial\Omega, \end{aligned}$$

were due to Caffarelli, Nirenberg and Spruck [4] for general curvature functions and Ivochkina [17] for the particular cases of higher order mean curvatures (1.2). These authors established the solvability of (1.3) for the case of uniformly convex domains and zero boundary values. In [18] Ivochkina extended her approach to embrace general boundary values and the more general k -convex domains, extending the result of J. Serrin [28] on the quasilinear case corresponding to the

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mean curvature. Despite the cases of higher order mean curvatures be covered by the generality allowed in the theorem of Caffarelli, Nirenberg and Spruck [4] their result makes use of a strong technical assumption on the curvature functions, which precludes the case of quotients $f = (S_{k,l})^{1/(k-l)}$, $0 \leq l < k \leq n$. However, the weak or viscosity solution approach by Trudinger [32] is sufficiently general in what concerns the curvatures functions so that it includes those quotient curvature functions. Also in [32] Trudinger establishes the existence theorems for Lipschitz solutions of (1.3) for general boundary values and domains subjects to natural geometric restrictions. In the subsequent articles [19] and [25] Ivochkina, Lin and Trudinger extended the approach used by Ivochkina in [18] to the cases of quotients, thereby obtaining globally smooth solutions. Their approach makes use of highly specific properties of these particular curvature functions. Finally, Sheng, Urbas and Wang [29] derive an interior curvature bound for solutions of (1.3) which permits to improve the existence results of Trudinger to yield locally smooth solutions.

For domains in a general Riemannian manifolds, the cases of mean, Gauss and harmonic curvatures have been extensively studied, as can be seen in [1], [2], [5], [9], [11], [12], [16], [20], [26], [27] and [31]. Nevertheless, to the best of our knowledge there are no results on the existence of solutions for (1.3) when f is a general curvature function (even for the case of higher order mean curvature) and Ω is a domain in a Riemannian manifold.

In this paper we deal with the Dirichlet problem (1.3) for general curvature functions in the general setting of domains Ω in a smooth Riemannian manifold (M^n, σ) under natural geometric conditions. We also derive a new boundary second derivative estimate which allows us to improve the existence results in [29], yielding globally smooth solutions. This is more precisely stated in what follows.

As in [4] and [29], we assume that $f \in C^2(\Gamma) \cap C^0(\bar{\Gamma})$ is a symmetric function defined in an open, convex, symmetric cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and containing the positive cone $\Gamma^+ = \{\kappa \in \mathbb{R}^n : \text{each component } \kappa_i > 0\}$. We suppose that f satisfies the fundamental structure conditions

$$(1.4) \quad f_i = \frac{\partial f}{\partial \kappa_i} > 0$$

and

$$(1.5) \quad f \text{ is a concave function.}$$

In addition, f is assumed to satisfy the following more technical assumptions

$$(1.6) \quad \sum f_i(\kappa) \geq c_0 > 0$$

$$(1.7) \quad \sum f_i(\kappa) \kappa_i \geq c_0 > 0$$

$$(1.8) \quad \limsup_{\kappa \rightarrow \partial\Gamma} f(\kappa) \leq \bar{\Psi}_0 < \Psi_0$$

$$(1.9) \quad f_i(\kappa) \geq c_0 > 0 \text{ for any } \kappa \in \Gamma \text{ with } \kappa_i < 0$$

$$(1.10) \quad (f_1 \cdots f_n)^{1/n} \geq c_0$$

for $\kappa \in \Gamma_\Psi = \{\kappa \in \Gamma : \Psi_0 \leq f(\kappa) \leq \Psi_1\}$ and a constant c_0 depending on Ψ_0 and Ψ_1 , where $\Psi_0 = \inf \Psi$ and $\Psi_1 = \sup \Psi$. In this context, a function $u \in C^2(\bar{\Omega})$ is called *admissible* if $\kappa[u] \in \Gamma$ at each point of its graph. We point out that these conditions also appear in [4], [10] and [32]. It has been shown (see [3], [4], [10], [22],

[24] and [32]) that the (root of) higher order mean curvatures and their quotients satisfy (1.4)-(1.10) on the appropriate cone Γ .

In [4] and [13], the hypotheses on the curvature functions include the requirement that for every constant $C > 0$ and every compact set E in Γ there is a number $R = R(C, E)$ such that

$$(1.11) \quad f(\kappa_1, \dots, \kappa_n + R) \geq C, \quad \forall \kappa \in E,$$

which precludes the important examples of the quotients $f = (S_{k,l})^{1/(k-l)}$. More precisely, this condition is used in [4] and [13] to estimate the double normal derivative of admissible solutions at the boundary. In this paper we adapt a technique presented in [32] to cover the cases where (1.11) does not hold.

As in the papers [4], [29], [32] we shall also need conditions on the boundary $\partial\Omega$ to ensure the attainment of Dirichlet boundary conditions. We assume that Ω is a domain with C^2 boundary and that the principal curvatures $\kappa' = (\kappa'_1, \dots, \kappa'_{n-1})$ of $\partial\Omega$ satisfy

$$(1.12) \quad f(\kappa'(y), 0) \geq \Psi(y, \varphi(y)), \quad \forall y \in \partial\Omega.$$

We note that (1.12) is the natural extension of the Serrin condition for the mean curvature case [28] and it implies that

$$(1.13) \quad (\kappa'_1, \dots, \kappa'_{n-1}, 0) \in \Gamma.$$

We can now formulate our main existence theorem for the Dirichlet problem. Let (M^n, σ) , $n \geq 2$, be a complete orientable Riemannian manifold and Ω a connected bounded domain in M .

Theorem 1. *Let $f \in C^2(\Gamma) \cap C^0(\overline{\Gamma})$ be a curvature function satisfying conditions (1.4)-(1.10). Suppose that there exists $0 < \alpha < 1$ such that $\partial\Omega \in C^{4,\alpha}$ and $\Psi \in C^{2,\alpha}(\overline{\Omega} \times \mathbb{R})$. Moreover, suppose that Ψ satisfies (1.12) and that $\Psi > 0$ and $\Psi_t \geq 0$ on $\overline{\Omega} \times \mathbb{R}$. If there exists a locally strictly convex C^2 function in $\overline{\Omega}$ and there exists an admissible subsolution $\underline{u} \in C^2(\overline{\Omega})$ of equation (1.1), then there exists a unique admissible solution $u \in C^{4,\alpha}(\overline{\Omega})$ of the Dirichlet problem (1.3) for any given function $\varphi \in C^{4,\alpha}(\overline{\Omega})$.*

The assumption on the existence of a strictly convex function in $C^2(\overline{\Omega})$ arises naturally when the problem is treated in a general Riemannian manifold, as could be seen for instance in [6] and [10]. Notice that when M is the Euclidean space this condition is always satisfied. Theorem 1 extends the result of Caffarelli, Nirenberg and Spruck presented in [4] to non-convex domains and general boundary values, without the assumption (1.11) and also improves the existence results of [29] and [32] to yield globally smooth solutions for general boundary values.

The cases $f = (S_{n,l})^{1/(n-l)}$, $l = 1, \dots, n-1$, are omitted from Theorem 1, as the corresponding extension of the Serrin condition (1.12) would imply $\Psi = 0$ on $\partial\Omega$, contradicting the hypothesis on the positivity of Ψ . However these cases are covered by Theorem 2 that is presented below.

It is well known that conditions on the geometry of the boundary $\partial\Omega$ play a key role in the study of the solvability of the Dirichlet problem (1.3). Nevertheless, several authors (e.g., [8], [9], [10], [11], [12] and [13]) had replaced geometric conditions on the boundary by assumptions on the existence of a subsolution satisfying the boundary condition. For more details we refer the reader to [3] and [10], where is shown the existence of a close relationship between the convexity of the boundary and the existence of such subsolutions. Therefore, it is natural to consider a

version of Theorem 1 obtained replacing the assumption on the geometry of the boundary $\partial\Omega$ by the assumption on the existence of a subsolution satisfying the boundary condition. In this context we obtain the following result.

Theorem 2. *Suppose that $f \in C^2(\Gamma) \cap C^0(\overline{\Gamma})$ satisfy (1.4)-(1.10) and that for some $0 < \alpha < 1$ it holds that*

- (i) $\partial\Omega \in C^{4,\alpha}$ has nonnegative mean curvature;
- (ii) there exists an admissible subsolution $\underline{u} \in C^2(\overline{\Omega})$ of equation (1.1) such that $\underline{u} = \varphi$ on $\partial\Omega$ and \underline{u} is locally strictly convex (up to the boundary) in a neighborhood of $\partial\Omega$;
- (iii) $\Psi \in C^{2,\alpha}(\overline{\Omega} \times \mathbb{R})$, $\Psi > 0$ and $\Psi_t \geq 0$ on $\overline{\Omega} \times \mathbb{R}$;
- (iv) there exists a locally strictly convex C^2 function in $\overline{\Omega}$.

Then there exists a unique admissible solution $u \in C^{4,\alpha}(\overline{\Omega})$ of the Dirichlet problem (1.3) for any given function $\varphi \in C^{4,\alpha}(\overline{\Omega})$.

Theorem 2 extends to domains in a general Riemannian manifold the result obtained by Guan and Spruck in [13]. In addition, Theorem 2 embraces a class of curvature functions larger than the one considered in [13], including the higher order mean curvatures and their quotients and, more generally, curvature functions that are defined in a general cone Γ , not necessarily being the positive cone Γ^+ . We point out that the requirement on the mean curvature of $\partial\Omega$ in (i) is necessary because of the generality of the cone Γ allowed in the Theorem 2. This kind of assumption was already made before in earlier contributions to the subject, see for instance [14]. Finally, the regularity assumption on \underline{u} in Theorems 1 and 2 can be weakened to $\underline{u} \in C^{0,1}(\overline{\Omega})$, provided \underline{u} is a subsolution in the viscosity sense. The proof proceeds as it will be presented below, with the usual comparison principle replaced by the comparison principle for viscosity solutions.

Following [4], [7], [13], [17] and [19], the proofs of the above existence theorems utilize the method of continuity which reduces the problem of the existence to the establishment of *a priori* estimates for a related family of Dirichlet problems in the Hölder space $C^{2,\beta}(\overline{\Omega})$ for some $\beta > 0$. Here we will establish C^2 *a priori* estimates. Hölder bounds for the second order derivatives then follows from the Evans-Krylov theory (see for example [7] and [21]) while higher order estimates follows from the classical Schauder theory. The uniqueness in Theorems 1 and 2 is a consequence of the comparison principle. As is pointed out in [4], [19], [20] and [25], the crucial estimates are those of the second derivatives on the boundary $\partial\Omega$. In this work, the use of a new barrier allows us to obtain estimates for mixed tangential normal derivatives at the boundary for solutions of (1.3) for general curvature equations, which is new even in the Euclidean case. This is one of the main achievements of this work. We establish the double normal second derivative estimates at the boundary by extending the techniques of [3], [10] and [33] to equations of curvature type.

This article is organized as follows: In Section 2 we list some basic formulas which are needed later. In Sections 3-6 we deal with the *a priori* estimates for prospective solutions of (1.3). The height and boundary gradient estimates are derived in Section 3. Section 4 is devoted to the proof of the global gradient estimates. In Section 5 *a priori* bounds for the second order derivatives on the boundary are established while in Section 6 we show how to estimate the second derivatives of solutions given boundary estimates for them. Finally, in Section 7 we complete the

proofs of Theorems 1 and 2 using the continuity method based on the previously established estimates.

2. PRELIMINARIES

Let (M^n, σ) be a complete Riemannian manifold. We consider the product manifold $\bar{M} = M \times \mathbb{R}$ endowed with the product metric. The Riemannian connections in \bar{M} and M will be denoted respectively by $\bar{\nabla}$ and ∇ . The curvature tensors in \bar{M} and M will be represented by \bar{R} and R , respectively. The convention used here for the curvature tensor is

$$R(U, V)W = \nabla_V \nabla_U W - \nabla_U \nabla_V W + \nabla_{[U, V]} W.$$

In terms of a coordinate system (x^i) we write

$$R_{ijkl} = \sigma \left(R \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right).$$

With this convention, the Ricci identity for the derivatives of a smooth function u is given by

$$(2.1) \quad u_{i;jk} = u_{i;kj} + R_{ilkj} u^l.$$

Let Ω be a bounded domain in M . Given a differentiable function $u : \Omega \rightarrow \mathbb{R}$, its graph is defined as the hypersurface Σ parameterized by $Y(x) = (x, u(x))$ with $x \in \Omega$. This graph is diffeomorphic with Ω and may be globally oriented by an unit normal vector field N for which it holds that $\langle N, \partial_t \rangle > 0$, where ∂_t denotes the usual coordinate vector field in \mathbb{R} . With respect to this orientation, the second fundamental form in Σ is by definition the symmetric tensor field $b = -\langle dN, dX \rangle$. We will denote by ∇' the connection of Σ .

The unit vector field

$$(2.2) \quad N = \frac{1}{W}(\partial_t - \nabla u)$$

is normal to Σ , where

$$(2.3) \quad W = \sqrt{1 + |\nabla u|^2}.$$

Here, $|\nabla u|^2 = u^i u_i$ is the squared norm of ∇u . The induced metric in Σ has components

$$(2.4) \quad g_{ij} = \langle Y_i, Y_j \rangle = \sigma_{ij} + u_i u_j$$

and its inverse has components given by

$$(2.5) \quad g^{ij} = \sigma^{ij} - \frac{1}{W^2} u^i u^j.$$

We easily verify that the components (a_{ij}) of the second fundamental form of Σ are determined by

$$a_{ij} = \langle \bar{\nabla}_{Y_j} Y_i, N \rangle = \frac{1}{W} u_{i;j}$$

where $u_{i;j}$ are the components of the Hessian $\nabla^2 u$ of u in Ω . Therefore the components a_i^j of the Weingarten map A^Σ of the graph Σ are given by

$$(2.6) \quad a_i^j = g^{jk} a_{ki} = \frac{1}{W} \left(\sigma^{jk} - \frac{1}{W^2} u^j u^k \right) u_{k;i}.$$

Above and throughout the text we made use of the Einstein summation convention.

For our purposes it is crucial to know the rules of commutation involving the covariant derivatives, the second fundamental form of a hypersurface and the curvature of the ambient. In this sense, the Gauss and Codazzi equations will play a fundamental role. They are, respectively,

$$(2.7) \quad R'_{ijkl} = \bar{R}_{ijkl} + a_{ik}a_{jl} - a_{il}a_{jk}$$

$$(2.8) \quad a_{ij;k} = a_{ik;j} + \bar{R}_{i0jk}$$

where the index 0 indicates coordinate components of the normal vector N and R' is the Riemann tensor of Σ . We note that $a_{ij;k}$ indicates the components of the tensor $\nabla' b$, obtained by differentiating covariantly the second fundamental form b of Σ with respect to the metric g . The following identity for commuting second derivatives of the second fundamental form will be quite useful. It was first found by Simons in [30] and in our notation it assumes the form

$$(2.9) \quad \begin{aligned} a_{ij;kl} = & a_{kl;ji} + a_{kl}a_i^m a_{jm} - a_{ik}a_j^m a_{lm} + a_{lj}a_i^m a_{km} - a_{ij}a_l^m a_{km} \\ & + \bar{R}_{likm}a_j^m + \bar{R}_{lijm}a_k^m - \bar{R}_{mjik}a_l^m - \bar{R}_{0i0j}a_{kl} + \bar{R}_{0l0k}a_{ij} \\ & - \bar{R}_{mkjl}a_i^m - \bar{\nabla}_l \bar{R}_{0jik} - \bar{\nabla}_i \bar{R}_{0kjl}. \end{aligned}$$

Let \mathcal{S} be the space of all symmetric covariant tensors of rank two defined in the Riemannian manifold (Σ, g) and \mathcal{S}_Γ be the open subset of those symmetric tensors $a \in \mathcal{S}$ for which the eigenvalues (with respect to the metric g) are contained in Γ . Then we can define the mapping $F : \mathcal{S}_\Gamma \rightarrow \mathbb{R}$ by setting

$$F(a) = f(\lambda(a)),$$

where $\lambda(a) = (\lambda_1, \dots, \lambda_n)$ are the eigenvalues of a . The mapping F is as smooth as f . Furthermore, F can be viewed as depending solely on the mixed tensor a^\sharp , obtained by raising one index of the given symmetric covariant 2-tensor a , as well as depending on the pair of covariant tensors (a, g) ,

$$F(a^\sharp) = F(a, g).$$

In terms of components, in an arbitrary coordinate system we have

$$F(a_i^j) = F(a_{ij}, g_{ij})$$

with $a_i^j = g^{jk}a_{ki}$. We denote the first derivatives of F by

$$F^{ij} = \frac{\partial F}{\partial a_{ij}} \quad \text{and} \quad F_i^j = \frac{\partial F}{\partial a_j^i},$$

and the second derivatives are indicated by

$$F^{ij,kl} = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}.$$

Hence F^{ij} are the components of a symmetric covariant tensor, while F_i^j defines a mixed tensor which is contravariant with respect to the index j and covariant with respect to the index i .

As in [18], we extend the cone Γ to the space of symmetric matrices of order n , which we denote (also) by \mathcal{S} . Namely, for $p \in \mathbb{R}^n$, let us define

$$\Gamma(p) = \{r \in \mathcal{S} : \lambda(p, r) \in \Gamma\},$$

where $\lambda(p, r)$ denotes the eigenvalues of the matrix $A(p, r) = g^{-1}(p)r$ given by

$$(2.10) \quad A(p, r) = \frac{1}{\sqrt{1 + |p|^2}} \left(I - \frac{p \otimes p}{1 + |p|^2} \right) r,$$

(the eigenvalues computed with respect to the Euclidean inner product). $A(p, r)$ is obtained from the matrix of the Weingarten map with (p, r) in place of $(\nabla u, \nabla^2 u)$ and δ^{ij} in place of σ^{ij} . We note that the eigenvalues of $A(p, r)$ are the eigenvalues of r (unless the $1/\sqrt{1 + |p|^2}$ factor) with respect to the inner product given by the matrix $g = I + p \otimes p$. In this setting it is convenient to introduce the notation

$$G(p, r) = F(A(p, r)) = f(\lambda(p, r)).$$

Hence, as in [4] and [13] we may write equation (1.1) in the form

$$(2.11) \quad F[u] = f(\kappa[u]) = G(\nabla u, \nabla^2 u) = \Psi(x, u).$$

In particular, if we denote

$$G^{ij} = \frac{\partial G}{\partial r_{ij}} \quad \text{and} \quad G^{ij,kl} = \frac{\partial^2 G}{\partial r_{ij} \partial r_{kl}},$$

we obtain

$$G^{ij} = \frac{1}{W} F^{ij} \quad \text{and} \quad G^{ij,kl} = \frac{1}{W^2} F^{ij,kl}.$$

The derivatives of the mapping F may be easily computed if we assume that the matrix (a_{ij}) is diagonal with respect to the metric (g_{ij}) , as is shown in the following lemma.

Lemma 3. *Let $\{e_i\}_{i=1}^n$ be a local orthonormal (with respect to the metric (g_{ij}) in Σ) basis of eigenvectors for $a \in \mathcal{S}_\Gamma$ with corresponding eigenvalues λ_i . Then, in terms of this basis the matrix (F^{ij}) is also diagonal with eigenvalues $f_i = \frac{\partial f}{\partial \lambda_i}$. Moreover, F is concave and its second derivatives are given by*

$$(2.12) \quad F^{ij,kl} \eta_{ij} \eta_{kl} = \sum_{k,l} f_{kl} \eta_{kk} \eta_{ll} + \sum_{k \neq l} \frac{f_k - f_l}{\lambda_k - \lambda_l} \eta_{kl}^2,$$

for any $(\eta_{ij}) \in \mathcal{S}$. Finally we have

$$(2.13) \quad \frac{f_i - f_j}{\lambda_i - \lambda_j} \leq 0.$$

These expressions must be interpreted as limits in the case of principal curvatures with multiplicity greater than one.

It follows from the above lemma that, under condition (1.4), equation (2.11) is elliptic, i.e., the matrix $G^{ij}(p, r)$ is positive-definite for any $r \in \Gamma(p)$. Moreover, under condition (1.5) the restriction of the function $G(p, \cdot)$ to the open set $\Gamma(p)$ is a concave function. We point out that since $1/W$ and 1 are respectively the lowest and the largest eigenvalues of g^{ij} it holds that

$$(2.14) \quad \frac{1}{W^3} F_i^j \delta_j^i \leq G^{ij} \delta_{ij} \leq \frac{1}{W} F_i^j \delta_j^i.$$

Now we analyze some consequences of the conditions (1.4)-(1.8). First we note that the concavity condition implies

$$(2.15) \quad \sum_i f_i(\kappa) \kappa_i \leq f$$

and we also may prove using assumptions (1.6)-(1.8) and following [3] that

$$(2.16) \quad \sum_i \kappa_i \geq \delta > 0,$$

for any $\kappa \in \Gamma$ that satisfies $f(\kappa) \geq \Psi_0$. This geometric fact implies that *upper bounds* for the principal curvatures of the graph of an admissible solution immediately ensure *lower bounds* for these curvatures.

Now we will derive a lemma that gives a useful formula involving the second and third derivatives of prospective solutions to the problem (1.3).

Lemma 4. *Let u be a solution of equation (2.11). The derivatives of u satisfy the formula*

$$(2.17) \quad \begin{aligned} G^{ij} u_{k;ij} = & W G^{ij} a_j^l u_{k;i} u_l + W G^{ij} a_j^l u_{k;l} u_i + \frac{1}{W} G^{jl} a_{jl} u^i u_{i;k} \\ & - G^{ij} R_{iljk} u^l + \Psi_k + \Psi_t u_k. \end{aligned}$$

Proof. Differentiating covariantly equation (2.11) in the k -th direction with respect to the metric σ of M we obtain

$$(2.18) \quad \Psi_k + \Psi_t u_k = \frac{\partial G}{\partial u_{i;j}} u_{i;jk} + \frac{\partial G}{\partial u_i} u_{i;k} = G^{ij} u_{i;jk} + G^i u_{i;k}.$$

From $F(a_i^j[u]) = G(\nabla u, \nabla^2 u)$ we calculate

$$\begin{aligned} G^i &= \frac{\partial G}{\partial u_i} = \frac{\partial F}{\partial a_r^s} \frac{\partial a_r^s}{\partial u_i} = F_s^r \frac{\partial}{\partial u_i} \left(\frac{1}{W} g^{sl} u_{l;r} \right) \\ &= F_s^r g^{sl} u_{l;r} \frac{\partial}{\partial u_i} \left(\frac{1}{W} \right) + \frac{1}{W} F_s^r \frac{\partial}{\partial u_i} (g^{sl}) u_{l;r}. \end{aligned}$$

We compute

$$F_s^r g^{sl} u_{l;r} \frac{\partial}{\partial u_i} \left(\frac{1}{W} \right) = -\frac{u^i}{W^3} F_s^r g^{sl} u_{l;r} = -\frac{1}{W} G^{rs} a_{rs} u^i$$

and

$$\begin{aligned} \frac{1}{W} F_s^r \frac{\partial}{\partial u_i} (g^{sl}) u_{l;r} &= G^{rp} g_{sp} \frac{\partial}{\partial u_i} (g^{sl}) u_{l;r} \\ &= -W G^{ij} a_j^l u_l - W G^{lj} a_l^i u_j, \end{aligned}$$

where we have used the expression

$$g_{sp} \frac{\partial g^{sl}}{\partial u_i} = -g^{sl} (\delta_{is} u_p + u_s \delta_{ip}) = -(\delta_{ip} g^{sl} u_s + g^{il} u_p).$$

It follows that

$$G^i = -\frac{1}{W} G^{rs} a_{rs} u^i - W G^{ij} a_j^l u_l - W G^{lj} a_l^i u_j.$$

Replacing these relations into (2.18) we obtain

$$\Psi_k + \Psi_t u_k = G^{ij} u_{i;jk} - \frac{1}{W} G^{rs} a_{rs} u^i u_{i;k} - W G^{ij} a_j^l u_l u_{i;k} - W G^{lj} a_l^i u_j u_{i;k}.$$

Using the Ricci identity (2.1), equation (2.17) is easily obtained. \square

A choice of an appropriate coordinate system simplifies substantially the computation of the components a_i^j of the Weingarten operator. We describe how to obtain such a coordinate system. Fixed a point $x \in M$, choose a geodesic coordinate system (x^i) of M around x such that the coordinate vectors $\{Y_* \cdot \frac{\partial}{\partial x^i} | x\}_{i=1}^n$ form a basis of principal directions of Σ at $Y(x)$ and $\{\frac{\partial}{\partial x^i} | x\}_{i=1}^n$ is an orthonormal basis with respect to the inner product given by the matrix $g = I + \nabla u \otimes \nabla u$. Hence,

$$a_i^j(x) = a_{ij}(x) = \frac{1}{W} u_{i;j}(x) \delta_{ij} = \kappa_i \delta_i^j$$

and

$$G^{ij} = \frac{1}{W} F_k^i g^{kj} = \frac{1}{W} f_i \delta_k^i \delta^{kj} = \frac{1}{W} f_i \delta_i^j$$

since (F_i^j) is diagonal whenever (a_i^j) is diagonal and $g^{ij} = \delta^{ij}$. From now on we refer to such a coordinate system as the *special* coordinate system centered at x .

At the center of a *special* coordinate system the formula (2.17) takes the simpler form

$$(2.19) \quad \begin{aligned} \sum_i f_i u_{k;ii} &= 2W \sum_i f_i \kappa_i u_{i;k} + \frac{1}{W} \sum_j f_j \kappa_j u^i u_{k;i} \\ &\quad - \sum_i f_i R_{ilik} u^l + W(\Psi_k + \Psi_t u_k). \end{aligned}$$

3. THE HEIGHT AND BOUNDARY GRADIENT ESTIMATES

In this section we start establishing the *a priori* estimates of admissible solutions of the Dirichlet problem (1.3). First we consider the Theorem 2. In this case the height estimate for admissible solutions is a direct consequence of the existence of a subsolution \underline{u} satisfying the boundary condition and of the inequality (2.16). In fact, it follows from the comparison principle applied to equation (1.3) that $\underline{u} \leq u$, which yields a lower bound. An upper bound is obtained using as barrier the solution \bar{u} of the Dirichlet problem

$$(3.1) \quad \begin{aligned} Q[\bar{u}] &= 0 & \text{in } \Omega \\ \bar{u} &= \varphi & \text{on } \partial\Omega, \end{aligned}$$

where Q is the mean curvature operator. The assumption on the geometry of $\partial\Omega$ ensures the existence of such a solution \bar{u} (see Theorem 1.5 in [31]). So, it follows from the comparison principle for quasilinear elliptic equations that $u \leq \bar{u}$. On the other hand, since $\underline{u} = u = \bar{u}$ on $\partial\Omega$, the inequality $\underline{u} \leq u \leq \bar{u}$ implies the boundary gradient estimate

$$|\nabla u| < C \quad \text{on } \partial\Omega.$$

Hence the height and the boundary gradient estimates are established in the case of Theorem 2. Now we consider the Theorem 1. First note that the assumption on the existence of a bounded subsolution and the solvability of (3.1) ensures the height estimates.

The boundary gradient estimate is obtained following closely the ideas presented in [32], which make use of the hypotheses on the boundary geometry to construct a lower barrier function. Indeed let d be the distance function to the

boundary $\partial\Omega$. In a small tubular neighborhood $\mathcal{N} = \{x \in \Omega : d(x) < a\}$ of $\partial\Omega$ we define the barriers in the form

$$(3.2) \quad w = \varphi - f(d),$$

where f is a suitable real function and $a > 0$ is a constant chosen sufficiently small to ensure that $d \in C^2(\bar{\mathcal{N}})$ (see [23]). The boundary function φ is redefined so that it is constant along normals to $\partial\Omega$ in \mathcal{N} and the function $f \in C^2([0, a])$ satisfies $f' > 0$ and $f'' < 0$. Fixed a point y_0 in \mathcal{N} , we fix around y_0 Fermi coordinates (y^i) in M along $\mathcal{N}_d = \{x \in \Omega : d(x) = d(y_0)\}$, such that y^n is the normal coordinate and the tangent coordinate vectors $\{\frac{\partial}{\partial y^\alpha}|_{y_0}\}$, $1 \leq \alpha \leq n-1$, form an orthonormal basis of eigenvectors that diagonalize $\nabla^2 d$ at y_0 . Since $\nabla d = \nu$ is the unit normal outward vector along \mathcal{N}_d we have

$$-\nabla^2 d(y_0) = \text{diag}(\kappa_1'', \kappa_2'', \dots, \kappa_{n-1}'', 0),$$

where $\kappa'' = (\kappa_1'', \kappa_2'', \dots, \kappa_{n-1}'')$ denotes the vector of principal curvatures of \mathcal{N}_d at y_0 . At y_0 we have $w_i = \varphi_i$ for $i < n$. Moreover $w_n(y_0) = \varphi_n - f'$ and

$$\nabla^2 w = \nabla^2 \varphi + \text{diag}(f' \kappa'', -f''),$$

since $d_n = 1$ and $d_i = 0$, $i < n$. Therefore, the matrix of the Weingarten operator of the graph of w at $(y_0, w(y_0))$ is

$$\begin{aligned} A[w] &= (g^{ik}(w)a_{jk}(w)) = \frac{1}{\sqrt{1+|\nabla w|^2}} \left(\delta^{jk} - \frac{w^j w^k}{1+|\nabla w|^2} \right) w_{k;j} \\ &= O\left(\frac{1}{v}\right) + \tilde{A}[w] \end{aligned}$$

as $v \rightarrow \infty$ (or equivalently $f' \rightarrow \infty$) where $v = \sqrt{1+|\nabla w|^2}$ and we have written

$$\tilde{A}[w] = \frac{1}{v} g^{-1}(w) \text{diag}(f' \kappa'', -f'').$$

It is convenient to split the computation of the matrix $\tilde{A}[w]$ into some blocks: For $i, j \leq n-1$ the components \tilde{a}_{ij} of $\tilde{A}[w]$ are

$$\tilde{a}_{ij} = \frac{1}{v} \left(\delta^{jk} - \frac{\varphi^j \varphi^k}{1+v^2} \right) f' \kappa_i'' \delta_{ik} = \kappa_i'' \delta_{ij} + O\left(\frac{1}{v}\right), \quad \text{as } v \rightarrow \infty.$$

For $i = n$ and $j < n$ we have

$$\tilde{a}_{nj} = -\frac{(f')^2 \kappa_j'' \varphi_j}{v^3} = O\left(\frac{1}{v}\right), \quad \text{as } v \rightarrow \infty,$$

and finally

$$\tilde{a}_{nn} = -f'' \frac{1+|\nabla \varphi|^2}{v^3}.$$

Now we take f of the form

$$f(d) = \frac{1}{b} \log(1+cd)$$

for positive constants $b, c > 0$ to be determined. We have

$$(3.3) \quad \begin{aligned} f'(d) &= \frac{c}{b(1+cd)} \geq \frac{1}{b(1+ca)} \\ f''(d) &= -bf'(d)^2, \end{aligned}$$

so

$$(3.4) \quad \tilde{a}_{nn} = b(f')^2 \frac{1 + |\nabla \varphi|^2}{v^3} = \frac{b}{v} \left(1 + O\left(\frac{1}{v}\right) \right), \quad \text{as } v \rightarrow \infty.$$

Hence the principal curvatures $\tilde{\kappa} = (\tilde{\kappa}_1, \dots, \tilde{\kappa}_n)$ of the graph of w at $(y_0, w(y_0))$ will differ from $\kappa_1'', \dots, \kappa_{n-1}'', \tilde{a}_{nn}$ by $O\left(\frac{1}{v}\right)$ as $v \rightarrow \infty$. Then it follows from (3.4) that we may estimate

$$\tilde{\kappa}_n \geq \frac{b}{2v}$$

provided $v \geq v_0$, $b \geq b_0$, where b_0 and v_0 are constants depending on $|\varphi|_2$ and $\partial\Omega$. Therefore

$$(3.5) \quad |\tilde{\kappa}_i - \kappa_i''| \leq \frac{b_1}{b} \tilde{\kappa}_n,$$

for a further constant b_1 . On the other hand, if $\tilde{y}_0 = \tilde{y}_0(y_0) \in \partial\Omega$ denotes the closest point in $\partial\Omega$ to y_0 we thus estimate

$$\begin{aligned} \Psi(y_0, w) &\leq \Psi(\tilde{y}_0, \varphi) + |\Psi|_1 d \\ &\leq \Psi(\tilde{y}_0, \varphi) + \frac{|\Psi|_1}{bv} \\ &\leq f(\kappa', 0) + \frac{|\Psi|_1}{bv}, \end{aligned}$$

where we used (3.3), the Serrin condition (1.12) and the assumption $\Psi_t \geq 0$. We recall that κ' denotes the principal curvatures of $\partial\Omega$. For $a > 0$ small, we can replace κ_i'' by κ_i' in (3.5). On the other hand, using the mean value theorem and conditions (1.4) and (1.13) we obtain positive constants δ_0, t_0 such that

$$(3.6) \quad f(\tilde{\kappa}) - f(\kappa', 0) \geq \delta_0 t \tilde{\kappa}_n$$

whenever $t \leq t_0$, $|\tilde{\kappa}_i - \kappa_i'| \leq t \tilde{\kappa}_n$, $i = 1, \dots, n-1$. To apply (3.6) we should observe that (1.4) and (1.12) imply $\tilde{\kappa} \in \Gamma$. Then, to deduce the inequality $F[w] \geq \Psi$ as desired we fix b so that

$$b \geq b_0, \frac{b_1}{t_0} \quad \text{and} \quad b^2 \geq \frac{|\Psi|_1}{\delta_0 t_0}.$$

Setting $M = \sup(\varphi - u)$ we then choose c and a in such a way that

$$ca = e^{bM} - 1 \quad \text{and} \quad c \geq v_0 b e^{bM}$$

to ensure $v \geq v_0$, $w \leq u$ on $\partial\mathcal{N}$. Therefore, we find that w is a lower barrier, that is,

$$\begin{aligned} F[w] &= f(\tilde{\kappa}[w]) > \Psi \quad \text{in } \mathcal{N} \\ w &\leq u \quad \text{on } \partial\mathcal{N}, \end{aligned}$$

which implies $u \geq w$ in \mathcal{N} . Since the condition (1.13) implies that the mean curvature of $\partial\Omega$ is nonnegative, we can conclude that there exists a solution \bar{u} of (3.1) which is an upper barrier. This establishes the boundary gradient estimates in the Theorem 1.

Remark 1. In the Lemma 7 below we use again the function w defined in (3.2). We note that if (1.13) holds, then for any $p \in \mathbb{R}^n$, we can choose the function w as above in such a way that it satisfies $\nabla^2 w \in \Gamma(p)$ on \mathcal{N} . To see this first note that the matrix $g^{-1}(p)$ has

eigenvalues $1/(1+|p|^2)^{3/2}$ and $1/\sqrt{1+|p|^2}$ with multiplicities 1 and $n-1$, respectively. Then the eigenvalues of the matrix $g^{-1}(p)\nabla^2 f(d)$ are

$$\tilde{\kappa} = \frac{1}{\sqrt{1+|p|^2}} \left(\tilde{\kappa}_1, \dots, \frac{\tilde{\kappa}_n}{1+|p|^2} \right),$$

where $\tilde{\kappa}' = (\tilde{\kappa}_1, \dots, \tilde{\kappa}_n)$ are the eigenvalues of $-\nabla^2 f(d)$. On the other hand, choosing f as above we conclude $\tilde{\kappa}' = f'(\kappa'_1, \dots, \kappa'_{n-1}, bf')$ on $\partial\Omega$, where $(\kappa'_1, \dots, \kappa'_{n-1})$ denote the principal curvatures of $\partial\Omega$. Hence, as the matrix $A(p, \nabla^2 w)$ has the form

$$A(p, \nabla^2 w) = g^{-1}(p)\nabla^2 w = g^{-1}(p)\nabla^2 \varphi - g^{-1}(p)\nabla^2 f(d),$$

it follows from (1.13) that for f' sufficiently large (depending on $|p|$, $|\varphi|_2$ and $\partial\Omega$) we have $\nabla^2 w \in \Gamma(p)$ on $\partial\Omega$. Since Γ is open, the same holds in a small tubular neighborhood N of $\partial\Omega$.

4. A PRIORI GRADIENT ESTIMATES

In this section we derive (the interior) *a priori* gradient estimates for admissible solutions u of the Dirichlet problem (1.3).

Proposition 5. *Let $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ be an admissible solution of (1.3). Then, under the conditions (1.4)-(1.9),*

$$(4.1) \quad |\nabla u| \leq C \quad \text{in } \overline{\Omega},$$

where C depends on $|u|_0$, $|\underline{u}|_1$ and other known data.

Proof. Set $\zeta(u) = ve^{Au}$, where $v = |\nabla u|^2 = u^k u_k$ and A is a positive constant to be chosen later. Let x_0 be a point where ζ attains its maximum. If $\zeta(x_0) = 0$ then $|\nabla u| = 0$ and so the result is trivial. If ζ achieves its maximum on $\partial\Omega$, then from the boundary gradient estimate obtained in the last section (4.1) holds and we are done. Hence, we are going to assume $x_0 \in \Omega$ and $\zeta(x_0) > 0$.

We fix a normal coordinate system (x^i) of M centered at x_0 , such that

$$\frac{\partial}{\partial x^1} \Big|_{x_0} = \frac{1}{|\nabla u|(x_0)} \nabla u(x_0).$$

In terms of these coordinates we have $u_1(x_0) = |\nabla u(x_0)| > 0$ and $u_j(x_0) = 0$ for $j > 1$. Since x_0 is a maximum for ζ ,

$$0 = \zeta_i(x_0) = 2e^{2Au(x_0)} (Avu_i(x_0) + u^l u_{l;i}(x_0))$$

and the matrix $\nabla^2 \zeta(x_0) = \{\zeta_{i;j}(x_0)\}$ is nonpositive. It follows that

$$(4.2) \quad u^l(x_0)u_{l;i}(x_0) = -Av(x_0)u_i(x_0)$$

for every $1 \leq i \leq n$. From now on all computations will be made at the point x_0 . As the matrix $\{G^{i;j}\}$ is positive definite one has

$$G^{ij}\zeta_{i;j} \leq 0.$$

We compute

$$\begin{aligned} \zeta_{i;j} = & 2e^{2Au} (u^l u_{l;i;j} + u^l_{;i} u_{l;j} + Avu_{i;j} + 2Au^l u_{l;j} u_i \\ & + 2Au^l u_{l;j} u_i + 2A^2 v u_i u_j). \end{aligned}$$

Hence

$$0 \geq \frac{1}{2e^{2Au}} G^{ij} \zeta_{i;j} = G^{ij} u^l u_{l;i;j} + G^{ij} u^l_{;i} u_{l;j} + Av G^{ij} u_{i;j} \\ + 4AG^{ij} u^l u_{l;j} u_j + 2A^2 v G^{ij} u_i u_j.$$

It follows from (4.2) that

$$4AG^{ij} u^l u_{l;i} u_j = -4A^2 v G^{ij} u_i u_j$$

and then

$$(4.3) \quad G^{ij} u^l u_{l;i;j} + G^{ij} u^l_{;i} u_{l;j} - 2A^2 v G^{ij} u_i u_j + Av G^{ij} u_{i;j} \leq 0.$$

We use the formula (2.17) at the Lemma 4 to obtain

$$G^{ij} u^l u_{l;i;j} = W G^{ij} a_j^k u^l u_{l;i} u_k + W G^{ij} a_j^k u^l u_{l;k} u_i + \frac{1}{W} G^{ij} a_{ij} u^l u^k u_{l;k} \\ - G^{ij} R_{iljk} u^l u^k + u^l \Psi_l + \Psi_t v.$$

Since

$$R_{ijlk} u^l u^k = 0$$

and

$$W G^{ij} a_j^k u^l u_{l;i} u_k = W G^{ij} a_j^k (-Av u_i) u_k = -Av W G^{ij} a_j^k u_i u_k$$

and

$$\frac{1}{W} G^{ij} a_{ij} u^l u^k u_{l;k} = \frac{1}{W} G^{ij} a_{ij} u^k (-Av u_k) = -\frac{1}{W} Av^2 G^{ij} a_{ij}$$

we get

$$G^{ij} u^l u_{l;i;j} = -2Av W G^{ij} a_j^k u_i u_k - \frac{Av^2}{W} G^{ij} a_{ij} + u^l \Psi_l + \Psi_t v.$$

Plugging this expression back into (4.3) we get

$$-2Av W G^{ij} a_j^k u_i u_k - \frac{Av^2}{W} G^{ij} a_{ij} + u^l \Psi_l + \Psi_t v \\ + G^{ij} u^l_{;i} u_{l;j} - 2A^2 v G^{ij} u_i u_j + Av G^{ij} u_{i;j} \leq 0.$$

Since $W a_{ij} = u_{i;j}$ we can rewrite the above inequality as

$$G^{ij} u^l_{;i} u_{l;j} - 2AW v G^{ij} a_j^k u_i u_k - 2A^2 v G^{ij} u_i u_j \\ + \left(Av W - \frac{Av^2}{W} \right) G^{ij} a_{ij} + u^l \Psi_l + \Psi_t v \leq 0.$$

Using the hypothesis $\Psi_t \geq 0$ we obtain

$$(4.4) \quad G^{ij} u^l_{;i} u_{l;j} - 2AW v G^{ij} a_j^k u_i u_k - 2A^2 v G^{ij} u_i u_j + \frac{Av}{W} G^{ij} a_{ij} + \Psi_l u^l \leq 0.$$

From the choice of the coordinate system and (4.2) it follows that

$$u_{1;1} = -Av \quad \text{and} \quad u_{1;i} = u_{i;1} = 0 \quad (i > 1).$$

After a rotation of the coordinates (x^2, \dots, x^n) we may assume that $\nabla^2 u = \{u_{i;j}(x_0)\}$ is diagonal. Since

$$a_i^j = g^{jk} a_{ki} = \frac{1}{W} \left(\sigma^{jk} - \frac{u^j u^k}{W^2} \right) u_{k;i},$$

at x_0 , we have

$$\begin{aligned} a_i^j &= 0 \quad (i \neq j) \\ a_1^1 &= \frac{1}{W^3} u_{1;1} = -\frac{Av}{W^3} < 0 \\ a_i^i &= \frac{1}{W} u_{i;i} \quad (i > 1). \end{aligned}$$

From Lemma 3, the matrix $\{F_i^j\}$ is diagonal. Then $\{G^{ij}\}$ is also diagonal with

$$G^{ii} = \frac{1}{W} F_k^i g^{ki} = \frac{1}{W} f_i \quad \text{and} \quad G^{11} = \frac{1}{W} F_k^1 g^{k1} = \frac{1}{W^3} f_1.$$

Using these relations and discarding the term

$$\frac{Av}{W} G^{ij} a_{ij} = \frac{Av}{W^2} \sum_i f_i \kappa_i \geq 0$$

we get from (4.4) the inequality

$$G^{ii} u_{i;i}^2 - 2AWvG^{11} a_1^1 (u_1)^2 - 2A^2 v G^{11} (u_1)^2 + \Psi_1 u_1 \leq 0,$$

which may be rewritten as

$$\sum_{\alpha > 1} G^{\alpha\alpha} u_{\alpha;\alpha}^2 + G^{11} \left(\frac{2A^2 v^3}{W^2} - 2A^2 v^2 + A^2 v^2 \right) + \Psi_1 \sqrt{v} \leq 0.$$

Since

$$\frac{2A^2 v^3}{W^2} - 2A^2 v^2 + A^2 v^2 = \frac{A^2 v^3 - A^2 v^2}{(1+v)^2}$$

we have

$$\sum_{\alpha > 1} G^{\alpha\alpha} u_{\alpha;\alpha}^2 + \frac{A^2 v^3 - A^2 v^2}{(1+v)^2} G^{11} + \Psi_1 \sqrt{v} \leq 0.$$

Then

$$\frac{A^2 v^3 - A^2 v^2}{(1+v)^2} \frac{1}{W^3} f_1 \leq -\Psi_1 \sqrt{v} \leq |D\Psi| \sqrt{v}.$$

Once

$$\kappa_1 = a_1^1 = -\frac{Av}{W^3} < 0,$$

we may apply hypothesis (1.9) to get $f_1 \geq c_0 > 0$, which implies

$$\frac{A^2 v^3 - A^2 v^2}{W^5 \sqrt{v}} \leq \frac{|D\Psi|}{c_0}.$$

Now we choose

$$A = \left(\frac{2}{c_0} \sup_{M \times I} |D\Psi| \right)^{1/2},$$

where I is the interval $I = [-C, C]$ with C being a uniform constant that satisfies $|u|_0 < C$. Therefore,

$$\frac{(u_1)^3 ((u_1)^2 - 1)}{(1 + (u_1)^2)^{5/2}} \leq \frac{1}{2},$$

that is,

$$(u_1)^5 - (u_1)^3 - \frac{1}{2} (1 + (u_1)^2)^{5/2} < 0.$$

Since $u_1 > 0$ this yields a bound for u_1 and hence for $\zeta(x_0)$, which implies the desired estimate. \square

5. BOUNDARY ESTIMATES FOR SECOND DERIVATIVES

In this section we establish the crucial *a priori* second derivatives estimates at the boundary. Bounds for pure tangential derivatives follow from the relation $u = \varphi$ on $\partial\Omega$. It remains to estimate the mixed and double normal derivatives.

Consider the linearized operator

$$L = G^{ij} - b^i,$$

where $b^i = \frac{1}{W^2} \sum_j f_j \kappa_j u^i$. It follows from (1.7), (2.15) and (4.1) that $|b^i| \leq C$ for a uniform constant C .

To proceed, we first derive some key preliminary lemmas. Let x_0 be a point on $\partial\Omega$. Let $\rho(x)$ denote the distance from x to x_0 , $\rho(x) = \text{dist}(x, x_0)$, and set

$$\Omega_\delta = \{x \in \Omega : \rho(x) < \delta\}.$$

Since $(\rho^2)_{i;j}(x_0) = 2\sigma_{ij}(x_0)$, by choosing $\delta > 0$ sufficiently small we may assume ρ smooth in Ω_δ and

$$(5.1) \quad \sigma_{ij} \leq (\rho^2)_{i;j} \leq 3\sigma_{ij} \quad \text{in } \Omega_\delta.$$

Since $\partial\Omega$ is smooth, we may also assume that the distance function $d(x)$ to the boundary $\partial\Omega$ is smooth in Ω_δ . In what follows, we redefine the boundary function φ in Ω_δ as being constant along the normals to $\partial\Omega$.

Let ξ be a C^2 arbitrary vector field defined in Ω_δ and η any extension to Ω_δ of the vector $\nabla u(x_0)$. Inspired in the approach used by Ivochkina in [18] we define the function

$$(5.2) \quad w = \langle \nabla u, \xi \rangle - \langle \nabla \varphi, \xi \rangle - \frac{1}{2} |\nabla u - \eta|^2.$$

The function w satisfies a fundamental inequality.

Proposition 6. *Assume that f satisfies (1.4)-(1.7). Then the function w satisfies*

$$(5.3) \quad L[w] \leq C(1 + G^{ij}\sigma_{ij} + G^{ij}w_i w_j) \quad \text{in } \Omega_\delta,$$

where C is a uniform positive constant.

Proof. For convenience we denote $\mu = \langle \nabla \varphi, \xi \rangle$. First we calculate the derivatives of w in an arbitrary coordinate system. We have

$$\begin{aligned} w_i &= \langle \nabla_i \nabla u, \xi \rangle + \langle \nabla u, \nabla_i \xi \rangle - \mu_i - \langle \nabla_i \nabla u - \nabla_i \eta, \nabla u - \eta \rangle \\ &= (\xi^k + \eta^k - u^k) u_{k;i} + ((\xi^k)_i + (\eta^k)_i) u_k - \mu_i - \langle \nabla_i \eta, \eta \rangle \end{aligned}$$

and

$$\begin{aligned} w_{i;j} &= \langle \nabla_j \nabla_i \nabla u, \xi \rangle + \langle \nabla_i \nabla u, \nabla_j \xi \rangle + \langle \nabla_j \nabla u, \nabla_i \xi \rangle + \langle \nabla u, \nabla_j \nabla_i \xi \rangle \\ &\quad - \mu_{i;j} - \langle \nabla_j \nabla_i \nabla u - \nabla_j \nabla_i \eta, \nabla u - \eta \rangle - \langle \nabla_i \nabla u - \nabla_i \eta, \nabla_j \nabla u - \nabla_j \eta \rangle \\ &= (\xi^k + K\xi^k - u^k) u_{k;i;j} + ((\xi^k)_j + (\eta^k)_j) u_{k;i} + ((\xi^k)_i + (\eta^k)_i) u_{k;j} \\ &\quad - u_{i;j}^k u_{k;j} + ((\xi^k)_{i;j} + (\eta^k)_{i;j}) u_k - \mu_{i;j} - \langle \nabla_i \eta, \nabla_j \eta \rangle - \langle \nabla_j \nabla_i \eta, \eta \rangle, \end{aligned}$$

where we denote by ξ^k , $(\xi^k)_i$ and $(\xi^k)_{i;j}$ the components of the vectors ξ , $\nabla_i \xi$ and $\nabla_j \nabla_i \xi$, respectively (the same notation is used for η). Therefore,

$$\begin{aligned} G^{ij} w_{i;j} &= (\xi^k + K \xi^k - u^k) G^{ij} u_{k;i;j} + 2G^{ij} ((\xi^k)_j + (\eta^k)_j) u_{k;i} - G^{ij} u_{;i}^k u_{k;j} \\ &\quad + G^{ij} ((\xi^k)_{i;j} + (\eta^k)_{i;j}) u_k - \mu_{i;j} - \langle \nabla_i \eta, \nabla_j \eta \rangle - \langle \nabla_j \nabla_i \eta, \eta \rangle. \end{aligned}$$

Now we use (2.17) to obtain

$$\begin{aligned} (\xi^k + \eta^k - u^k) G^{ij} u_{k;i;j} &= W (\xi^k + \eta^k - u^k) G^{ij} a_j^l u_{k;i} u_l + W (\xi^k + \eta^k - u^k) \\ &\quad \times G^{ij} a_j^l u_{k;l} u_i + \frac{1}{W} (\xi^k + \eta^k - u^k) G^{jl} a_{jl} u^i u_{k;i} \\ &\quad + (\xi^k + \eta^k - u^k) (\Psi_k + \Psi_t u_k - G^{ij} R_{iljk} u^l). \end{aligned}$$

On the other hand, it follows from the expression for w_i that

$$(\xi^k + \eta^k - u^k) u_{i;k} = w_i - ((\xi^k)_i + (\eta^k)_i) u_k + \mu_i + \langle \nabla_i \eta, \eta \rangle.$$

Substituting this equality in the above equations we get

$$\begin{aligned} (5.4) \quad G^{ij} w_{i;j} &= W G^{ij} a_j^l w_i u_l + W G^{ij} a_j^l w_l u_i + \frac{1}{W} G^{jl} a_{jl} u^i w_i - G^{ij} u_{;i}^k u_{k;j} \\ &\quad + 2G^{ij} ((\xi^k)_j + (\eta^k)_j) u_{k;i} + W G^{ij} a_j^l u_l (\mu_i - ((\xi^k)_i + (\eta^k)_i) u_k \\ &\quad + \langle \nabla_i \eta, \eta \rangle) + W G^{ij} a_j^l u_i (\mu_l - ((\xi^k)_l + (\eta^k)_l) u_k + \langle \nabla_l \eta, \eta \rangle) + \\ &\quad \frac{1}{W} G^{jl} a_{jl} u^i (\mu_i - ((\xi^k)_i + (\eta^k)_i) u_k + \langle \nabla_i \eta, \eta \rangle) \\ &\quad + G^{ij} ((\xi^k)_{i;j} + (\eta^k)_{i;j}) u_k - \mu_{i;j} - \langle \nabla_i \eta, \nabla_j \eta \rangle - \langle \nabla_j \nabla_i \eta, \eta \rangle \\ &\quad - (\xi^k + \eta^k - u^k) R_{iljk} u^l + (\xi^k + \eta^k - u^k) (\Psi_k + \Psi_t u_k). \end{aligned}$$

Since (5.3) does not depend on the coordinate system, i.e., it is a tensorial inequality, it is sufficient to prove it in a fixed coordinate system. Given $x \in \Omega$, let (x^i) be the *special* coordinate system centered at x . In terms of this coordinates the inequality (5.3) takes at x the form

$$(5.5) \quad L[w] = \frac{1}{W} \sum_i f_i w_{i;i} - b^i w_i \leq C \left(1 + \frac{1}{W} \sum_i f_i \sigma_{ii} + \frac{1}{W} \sum_i f_i w_i^2 \right).$$

We will prove the above inequality. In what follows all computations are done at the point x .

In these coordinates we have $\kappa_i = a_i^j = a_{ij} = \frac{1}{W} u_{i;j} \delta_{ij}$ and $G^{ij} = \frac{1}{W} f_i \delta_i^j$. Since the quantities depending on ∇u , ξ , η and μ are under control, we get

$$\begin{aligned}
WG^{ij} a_j^l w_i u_l &= \sum_i f_i \kappa_i w_i u_i \leq \varepsilon \sum_i f_i \kappa_i^2 + \frac{1}{\varepsilon} \sum_i f_i w_i^2 u_i^2 \\
&\leq \varepsilon \sum_i f_i \kappa_i^2 + C \sum_i f_i w_i^2 \\
2G^{ij} ((\xi^k)_j + (\eta^k)_j) u_{k;i} &= 2((\xi^i)_i + (\eta^i)_i) f_i \kappa_i \leq \varepsilon \sum_i f_i \kappa_i^2 + C \sum_i f_i \\
WG^{ij} a_j^l \mu_i &= \sum_i f_i \kappa_i u_i \mu_i \leq \varepsilon \sum_i f_i \kappa_i^2 + C \sum_i f_i \\
G^{ij} u_{i;k}^k u_{k;j} &= G^{ij} \sigma^{kl} u_{l;i} u_{k;j} = W^2 G^{ij} \sigma^{kl} a_{l;i} a_{k;j} = W \sigma^{ii} f_i \kappa_i^2 \\
&\geq C_0 \sum_i f_i \kappa_i^2 \\
\frac{1}{W} G^{jl} a_{jl} u^i \mu_i &= \frac{1}{W^2} \sum_j f_j \kappa_j u^i \mu_i \leq C \sum_j f_j \kappa_j \leq C \\
G^{i;j} \mu_{i;j} &= \frac{1}{W} \sum_i f_i \mu_{i;i} \leq C \sum_i f_i,
\end{aligned}$$

where $\varepsilon > 0$ is any positive number and $C_0 > 0$ depends only on $\sigma|_{\overline{\Omega}}$. To obtain the above inequalities we made use of the ellipticity condition $f_i > 0$. Estimating all the terms in (5.4) as above, we conclude that equality (5.4) implies

$$(5.6) \quad L[w] \leq (\varepsilon C - C_0) \sum_i f_i \kappa_i^2 + C \sum_i f_i w_i^2 + C \sum_i f_i + C.$$

Choosing $\varepsilon > 0$ sufficiently small such that the first term on the sum above becomes negative we obtain

$$L[w] \leq C(1 + \sum_i f_i + \sum_i f_i w_i^2).$$

Using that $\sigma_{ii} > C_0 > 0$ in $\overline{\Omega}$ and W is under control, we get (5.5). \square

We note that inequality (5.3) may be simplified further. In fact, since

$$G^{ij} \sigma_{ij} \geq \delta_0 > 0,$$

replacing C to $C/\delta_0 + C$ (we may assume $1 > \delta_0 > 0$) we get

$$(5.7) \quad L[w] \leq C(G^{ij} \sigma_{ij} + G^{ij} w_i w_j) \quad \text{in } \Omega_\delta.$$

Setting

$$(5.8) \quad \tilde{w} = 1 - e^{-a_0 w}$$

for a positive constant a_0 , we get $\tilde{w}_i = a_0 e^{-a_0 w} w_i$ and $\tilde{w}_{i;j} = a_0 e^{-a_0 w} (w_{i;j} - a_0 w_i w_j)$. Therefore,

$$L[\tilde{w}] = G^{ij} \tilde{w}_{i;j} - b^i \tilde{w}_i = a_0 e^{-a_0 w} (L[w] - a_0 G^{ij} w_i w_j),$$

if we choose a_0 large such that $a_0 \geq C$, where C is the constant in (5.7),

$$L[w] - a_0 G^{ij} w_i w_j \leq L[\tilde{w}] - C G^{ij} w_i w_j \leq C G^{ij} \sigma_{ij}.$$

Hence

$$(5.9) \quad L[\tilde{w}] \leq CG^{ij}\sigma_{ij}.$$

Now we extend to curvature equations the Lemma 6.2 in [10] obtained by Guan to Hessian equations. This lemma gives the elements to complete the construction of a barrier function.

Lemma 7. *Assume that f satisfies (1.4)-(1.10). Then there exist some uniform positive constants t, δ, ε sufficiently small and N sufficiently large such that the function*

$$(5.10) \quad v = (u - \underline{u}) + td - \frac{N}{2}d^2$$

satisfies

$$(5.11) \quad L[v] \leq -\varepsilon(1 + G^{ij}\sigma_{ij}) \quad \text{in } \Omega_\delta$$

and

$$v \geq 0 \quad \text{on } \partial\Omega_\delta.$$

Proof. Since \underline{u} is locally strictly convex in a neighborhood of $\partial\Omega$ we may choose $\delta > 0$ small enough such that the eigenvalues $\lambda(\nabla^2 \underline{u}) \in \Gamma^+$ in Ω_δ . In particular, we have $\nabla^2 \underline{u} \in \Gamma(\nabla u)$ in Ω_δ . Consider the function $v^* = \underline{u} - 3\varepsilon\rho^2$. Since $\Gamma(\nabla u)$ is open and $F[\underline{u}] > 0$, we may choose $\varepsilon > 0$ sufficiently small, such that v^* is admissible and $\nabla^2 v^* \in \Gamma(\nabla u)$ in Ω_δ .

It follows from the concavity of $G(p, \cdot)$ that

$$G^{ij}(p, r)(r_{ij} - s_{ij}) \leq G(p, r) - G(p, s) \quad \text{for all } r, s \in \Gamma(p).$$

Applying this inequality we get

$$\begin{aligned} L[u - \underline{u}] &= L[u - v^* - 3\varepsilon\rho^2] \\ &= G^{ij}(u_{i;j} - v_{i;j}^*) - b^i(u_i - v_i^*) - 3\varepsilon L[\rho^2] \\ &\leq G(\nabla u, \nabla^2 u) - G(\nabla u, \nabla^2 v^*) - b^i(u_i - v_i^*) \\ &\quad - 3\varepsilon G^{ij}(\rho^2)_{i;j} + 6\varepsilon \rho b^i \rho_i. \end{aligned}$$

Since $G(\nabla u, \nabla^2 u) = \Psi$ and $G(\nabla u, \nabla^2 v^*) > 0$, it follows from the C^1 estimate and the boundedness of b^i that

$$L[u - \underline{u}] \leq C_1 - 3\varepsilon G^{ij}(\rho^2)_{i;j}.$$

Hence, we conclude from (5.1)

$$(5.12) \quad L[u - \underline{u}] \leq C_1 - 3\varepsilon G^{ij}\sigma_{i;j}.$$

As in the previous lemma, the inequality proposed is a tensorial one. So, it is sufficient to prove (5.11) in a fixed coordinate system. For $\delta > 0$ small we may define Fermi coordinates (y^i) on Ω_δ along $\partial\Omega$, such that $y^n = d$ is the normal coordinate. In these coordinates we have $d_\alpha = 0, 1 \leq \alpha \leq n-1$, and $d_n = 1$. Hence, a straightforward computation yields

$$L\left[td - \frac{N}{2}d^2\right] = (t - dN)L[d] - NG^{nn}.$$

Since there exists a uniform positive constant C that satisfies $d_{i,j} \leq C\sigma_{ij}$ in Ω_δ and $|b^i| < C$, we have

$$L \left[td - \frac{N}{2}d^2 \right] \leq C_2(t + N\delta)(1 + G^{ij}\sigma_{ij}) - NG^{nn}.$$

This inequality and (5.12) give

$$\begin{aligned} L[v] &\leq L[u - \underline{u}] + L \left[td - \frac{N}{2}d^2 \right] \\ &\leq C_1 - 3\varepsilon G^{ij}\sigma_{ij} + C_2(t + N\delta)(1 + G^{ij}\sigma_{ij}) - NG^{nn} \\ &= C_1 + C_2(t + N\delta) + (C_2(t + N\delta) - 3\varepsilon) G^{ij}\sigma_{ij} - NG^{nn}. \end{aligned}$$

As in [10], we choose indices such that $f_1 \geq \dots \geq f_n$. Since the eigenvalues of the matrix G^{ij} are $\frac{1}{W}f_1, \dots, \frac{1}{W}f_n$, it follows from our choice of indices that

$$G^{nn} \geq \frac{1}{W}f_n \geq c_1f_n \quad \text{and} \quad G^{ij}\sigma_{ij} \geq c_2 \sum_i f_i.$$

Using the arithmetic-geometric mean inequality and (1.10) we get

$$\begin{aligned} \varepsilon G^{ij}\sigma_{ij} + NG^{nn} &\geq c_2 \sum_i f_i + c_1 N f_n \\ &\geq cn\varepsilon(Nf_1 \cdot \dots \cdot f_n)^{1/n} = C_3 N^{1/n}. \end{aligned}$$

Now we apply this relation into the above inequality to get

$$L[v] \leq C_1 + C_2(t + N\delta) + (C_2(t + N\delta) - 2\varepsilon)G^{ij}\sigma_{ij} - C_3 N^{1/n}.$$

Since $\delta^2 \leq t\delta/N$ implies $t\delta - N/2\delta^2 \geq 0$ and $u \geq \underline{u}$, we choose $t = \frac{\varepsilon}{2C_2}$ and $\delta \leq \frac{t}{N}$ to get $v \geq 0$ on $\Omega \cap \partial\Omega_\delta$. With this choice we have

$$L[v] \leq C_1 - \varepsilon G^{ij}\sigma_{ij} - C_3 N^{1/n}.$$

By choosing N large such that $C_3 N^{1/n} \geq C_1 + 2\varepsilon$ we obtain (5.11). \square

Remark 2. Under the hypotheses of Theorem 1 we construct a subsolution w defined in Ω_δ and that is not necessarily strictly convex but satisfies $\nabla^2 w \in \Gamma(\nabla u)$. We replace \underline{u} by w in the Lemma above to get the result. See Remark 1.

5.1. Mixed Second Derivative Boundary Estimate. We define the function

$$(5.13) \quad h = \tilde{w} + b_0\rho^2 + c_0v,$$

where b_0 and c_0 are constants to be chosen later. Assume the vector field ξ is tangent along $\partial\Omega \cap \partial\Omega_\delta$. Hence

$$\tilde{w} = 1 - \exp \left(a_0 \frac{1}{2} |\nabla u - \eta|^2 \right)$$

on $\partial\Omega \cap \partial\Omega_\delta$. Since $\eta(x_0) = \nabla u(x_0)$, for any vector field η tangent along $\partial\Omega \cap \partial\Omega_\delta$, we have $\tilde{w}(x_0) = \nabla_\eta \tilde{w}(x_0) = 0$. Hence we conclude that $\tilde{w} = O(\rho^2)$ on $\partial\Omega \cap \partial\Omega_\delta$, if $\delta > 0$ is small enough. Then, since $v \geq 0$ on $\partial\Omega_\delta$, if b_0 is sufficiently large we have $h \geq 0$ on $\partial\Omega_\delta$. On the other hand, it follows from (5.1), (5.9) and (5.11) that

$$\begin{aligned} L[h] &= L[\tilde{w}] + b_0 L[\rho^2] + c_0 L[v] \\ &\leq (C_1 + C_2 b_0 - c_0 \varepsilon)(1 + G^{ij}\sigma_{ij}) + b_0. \end{aligned}$$

Therefore, for $c_0 \gg b_0 \gg 1$ both sufficiently large, we get $L[h] \leq 0$ in Ω_δ and $h \geq 0$ on $\partial\Omega_\delta$. It follows from the maximum principle that $h \geq 0$ in Ω_δ . Consequently,

$$\nabla_\nu h(x_0) \geq 0,$$

which give us

$$u_{\xi;\nu}(x_0) \geq -\langle \nabla u, \nabla_\nu \xi \rangle(x_0) - \frac{c_0}{a_0}(u - \underline{u})_\nu(x_0) - \frac{c_0}{a_0}t.$$

Replacing ξ by $-\xi$ at the definition of w we establish a bound for the mixed normal-tangential derivatives on $\partial\Omega$

$$|u_{\xi;\nu}(x_0)| \leq C,$$

for any direction tangent ξ to $\partial\Omega$. Since x_0 is arbitrary, we have

$$(5.14) \quad |u_{\xi;\nu}| < C \quad \text{on } \partial\Omega.$$

5.2. Double Normal Second Derivative Boundary Estimate. For the pure normal second derivative, since $\sum_i \kappa_i[u] \geq \delta_0 > 0$, we need only to derive an upper bound

$$(5.15) \quad u_{\nu;\nu} \leq C \quad \text{on } \partial\Omega.$$

First we note that the equality $u = \varphi$ on $\partial\Omega$ implies

$$(5.16) \quad u_{\xi;\eta}(y) = \varphi_{\xi;\eta}(y) - u_\nu(y)\Pi(\xi, \eta)(y),$$

for any tangent vectors $\xi, \eta \in T_y(\partial\Omega) \subset T_y M$, $y \in \partial\Omega$, where Π denotes the second fundamental form of $\partial\Omega$. Let T_u be the $(0, 2)$ tensor defined on $\partial\Omega$ by

$$(5.17) \quad T_u = (\tilde{\nabla}^2 \varphi - u_\nu \Pi),$$

where $\tilde{\nabla}$ is the induced connection on $\partial\Omega$. Since $a_{\alpha\beta} = \frac{1}{W}u_{\alpha;\beta}$, it follows from the equality (5.16) that the components of T_u in terms of coordinates (y^α) in $\partial\Omega$ are $W a_{\alpha\beta}$. We denote by $\tilde{\kappa} = (\tilde{\kappa}_1, \dots, \tilde{\kappa}_{n-1})$ the eigenvalues of the tensor T_u with respect to the inner product defined on $\partial\Omega$ by the matrix $\tilde{g} = \tilde{\sigma} + \tilde{\nabla}\varphi \otimes \tilde{\nabla}\varphi$, where $\tilde{\sigma}$ is the metric on $\partial\Omega$ induced by σ .

Let Γ' be the projection of Γ on \mathbb{R}^{n-1} . We denote by $d(\kappa')$ the distance from $\kappa' \in \Gamma'$ to $\partial\Gamma'$. We point out that Γ' is also an open convex symmetric cone.

We are going to analyze the behavior of $d(\kappa'[u])$ for admissible solutions u . First we fix Fermi coordinates (y^i) in M along $\partial\Omega$, such that y^n is the normal coordinate and the tangent coordinate vectors $\{\frac{\partial}{\partial y^\alpha}|_{y_0}\}$, $1 \leq \alpha \leq n-1$, is an orthonormal basis of eigenvectors that diagonalize T_u at a given $y_0 \in \partial\Omega$, with respect to the inner product $\tilde{g} = \tilde{\sigma} + \tilde{\nabla}\varphi \otimes \tilde{\nabla}\varphi$. At y_0 the matrix of the second fundamental form of Σ in terms of this coordinate system is given by

$$(5.18) \quad a_{ij} = \frac{1}{W}u_{i;j}\delta_{ij}, \quad (i, j < n), \quad a_{in} = \frac{1}{W}u_{\nu;i}, \quad (i < n), \quad a_{nn} = \frac{1}{W}u_{\nu;\nu}.$$

It follows from (5.16) that $\tilde{\kappa} = (u_{1;1}, \dots, u_{n-1;n-1})$ are also the eigenvalues of the tensor T_u defined above. Since the principal curvatures $\kappa[u] = (\kappa_1, \dots, \kappa_n)$ of Σ at $(y_0, u(y_0))$ are the roots of the equation $\det(a_{ij} - \kappa g_{ij}) = 0$ and $g_{\alpha\beta}(y_0) =$

$\tilde{g}_{\alpha\beta}(y_0) = \delta_{\alpha\beta}$ for $1 \leq \alpha, \beta \leq n-1$, they satisfy

$$\det \begin{pmatrix} \frac{1}{W}u_{1;1} - \kappa & 0 & \cdots & \frac{1}{W}u_{1;\nu} - g_{1n} \\ 0 & \frac{1}{W}u_{2;2} - \kappa & \cdots & \frac{1}{W}u_{2;\nu} - g_{2n} \\ \vdots & & \ddots & \vdots \\ \frac{1}{W}u_{\nu;1} - g_{1n} & \frac{1}{W}u_{\nu;1} - g_{2n} & \cdots & \frac{1}{W}u_{\nu;\nu} - \kappa g_{n;n} \end{pmatrix} = 0.$$

Therefore, by Lemma 1.2 of [3] the principal curvatures $\kappa[u](y) = (\kappa_1, \dots, \kappa_n)$ of Σ , at $(y_0, u(y_0))$, behave like

$$(5.19) \quad \kappa_\alpha = \frac{1}{W}u_{\alpha;\alpha} + o(1), \quad 1 \leq \alpha \leq n-1,$$

$$(5.20) \quad \kappa_n = \frac{1}{Wg_{n;n}}u_{\nu;\nu} \left(1 + O\left(\frac{1}{u_{\nu;\nu}} \right) \right),$$

as $|u_{\nu;\nu}| \rightarrow \infty$. Since u is admissible, we have $\kappa'[u] = (\kappa_\alpha) \in \Gamma'$, therefore $W\kappa'[u] \in \Gamma'$. Hence, for $u_{\nu;\nu}$ large we have $\tilde{\kappa} = (u_{1;1}, \dots, u_{n-1;n-1}) \in \Gamma'$, since Γ' is open and we can assume $u_{\nu;\nu} \geq 0$. Since $y_0 \in \partial\Omega$ is arbitrary, it follows from the gradient, tangent and tangent-normal second derivative estimates previously established that there exists a uniform positive constant $N_0 > 0$ such that the eigenvalues $\tilde{\kappa}$ of T_u satisfy

$$(5.21) \quad \tilde{\kappa} \in \Gamma', \quad \text{if } u_{\nu;\nu} \geq N_0.$$

The following lemma is the key ingredient to obtain the double normal boundary estimate. It is an adaption of the technique used in [3] and [10], using the brilliant idea introduced by Trudinger in [33].

Lemma 8. *Let $N_0 > 0$ be the constant defined in (5.21) and suppose $u_{\nu;\nu} \geq N_0$. Then there exists a uniform constant $c_0 > 0$ such that*

$$d(y) = d(\tilde{\kappa}[u](y)) \geq c_0 \quad \text{on } \partial\Omega.$$

Proof. Consider a point $y_0 \in \partial\Omega$ where the function $d(y)$ attains its minimum in $\partial\Omega$. It suffices to prove that $d(y_0) \geq c_0 > 0$. As above we fix Fermi coordinates (y^i) in M along $\partial\Omega$, centered at y_0 , such that y^n is the normal coordinate and the tangent coordinate vectors $\{\frac{\partial}{\partial y^\alpha}|_{y_0}\}_{\alpha < n}$ diagonalize T_u at y_0 with respect to the inner product given by $\tilde{\sigma} + \tilde{\nabla}\varphi \otimes \tilde{\nabla}\varphi$. We choose indices such that

$$\tilde{\kappa}_1(y_0) \leq \cdots \leq \tilde{\kappa}_{n-1}(y_0).$$

From (5.16) the coordinate system (y^α) diagonalizes also the restriction of $\nabla^2 u$ to $T(\partial\Omega)$ at y_0 and

$$(5.22) \quad \tilde{\kappa}_\alpha(y_0) = u_{\alpha;\alpha}(y_0), \quad \alpha < n.$$

We extend ν to the coordinate neighborhood by taking its parallel transport along normal geodesics departing from $\partial\Omega$ and set

$$b_{\alpha\beta} = \Pi \left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \right) = \left\langle \nabla_{\frac{\partial}{\partial y^\alpha}} \frac{\partial}{\partial y^\beta}, \nu \right\rangle.$$

Using Lemma 6.1 of [3], we find a vector $\gamma' = (\gamma_1, \dots, \gamma_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$\gamma_1 \geq \cdots \geq \gamma_{n-1} \geq 0, \quad \sum_{\alpha < n} \gamma_\alpha = 1$$

and

$$(5.23) \quad d(y_0) = \sum_{\alpha < n} \gamma_\alpha \tilde{\kappa}_\alpha(y_0) = \sum_{\alpha < n} \gamma_\alpha u_{\alpha;\alpha}(y_0).$$

Moreover,

$$(5.24) \quad \Gamma' \subset \{\lambda' \in \mathbb{R}^{n-1} : \gamma' \cdot \lambda' > 0\}.$$

Applying Lemma 6.2 of [3], with $\gamma_n = 0$, we get for all $y \in \partial\Omega$ near y_0

$$(5.25) \quad \sum_{\alpha < n} \gamma_\alpha T_{\alpha\alpha}(y) = \sum_{\alpha < n} \gamma_\alpha u_{\alpha;\alpha}(y) \geq \sum_{\alpha < n} \gamma_\alpha \tilde{\kappa}_\alpha(y) \geq d(y) \geq d(y_0),$$

where we have used (5.33) and $|\gamma| \leq 1$ in the second inequality. Differentiating covariantly the equality $u - \varphi = 0$ on $\partial\Omega$ we get

$$(5.26) \quad (u - \varphi)_{\xi;\eta} = -(u - \varphi)_\nu \Pi(\xi, \eta) \quad \text{on } \partial\Omega,$$

for any vectors fields ξ and η tangent to $\partial\Omega$. Then, for $y \in \partial\Omega$ near y_0 , we have

$$u_\nu(y) \sum_{\alpha < n} \gamma_\alpha b_{\alpha\alpha}(y) = \sum_{\alpha < n} \gamma_\alpha (\varphi - u)_{\alpha;\alpha}(y).$$

Then

$$(5.27) \quad \begin{aligned} u_\nu(y) \sum_{\alpha < n} \gamma_\alpha b_{\alpha\alpha}(y) &= \sum_{\alpha < n} \gamma_\alpha \varphi_{\alpha;\alpha}(y) - \sum_{\alpha < n} \gamma_\alpha u_{\alpha;\alpha}(y) \\ &\leq \sum_{\alpha < n} \gamma_\alpha \varphi_{\alpha;\alpha}(y) - d(y_0), \end{aligned}$$

where we use (5.34) in the last inequality.

Since \underline{u} is locally strictly convex in a neighborhood of $\partial\Omega$ it follows that $\kappa'(\underline{u}_{\alpha;\beta}(y_0))$ belongs to Γ' (since $\Gamma^+ \subset \Gamma$). We point out that $\kappa'(\underline{u}_{\alpha;\beta})$ denotes the eigenvalues of $\nabla^2 \underline{u}$. We may assume

$$d(y_0) < \frac{1}{2} d(\kappa'(\underline{u}_{\alpha;\beta}(y_0))),$$

otherwise we are done. Now we use the equality $u = \underline{u}$ on $\partial\Omega$ to get

$$(u - \underline{u})_\nu \sum_{\alpha < n} \gamma_\alpha b_{\alpha\alpha} = \sum_{\alpha < n} \gamma_\alpha (\underline{u} - u)_{\alpha;\alpha},$$

on $\partial\Omega$. Therefore we conclude from (5.35), (5.33) and Lemma 6.2 of [3] that

$$\begin{aligned} (u - \underline{u})_\nu(y_0) \sum_{\alpha < n} \gamma_\alpha b_{\alpha\alpha}(y_0) &= \sum_{\alpha < n} \gamma_\alpha \underline{u}_{\alpha;\alpha}(y_0) - \sum_{\alpha < n} \gamma_\alpha u_{\alpha;\alpha}(y_0) \\ &\geq d(\kappa'(\underline{u}_{\alpha;\beta}(y_0))) - d(y_0) > \frac{1}{2} d(\underline{u}_{\alpha;\beta}(y_0)) > 0. \end{aligned}$$

Since $(u - \underline{u})_\nu \geq 0$ on $\partial\Omega$, there exist uniform positive constants $c, \delta > 0$, such that

$$\sum_{\alpha < n} \gamma_\alpha b_{\alpha\alpha}(y) \geq c > 0,$$

for every $y \in \Omega$ satisfying $\text{dist}(y, y_0) < \delta$. Hence we may define the function

$$(5.28) \quad \mu(y) = \frac{1}{\sum_{\alpha < n} \mu_\alpha b_{\alpha\alpha}(y)} \left(\sum_{\alpha < n} \mu_\alpha \varphi_{\alpha;\alpha}(y) - d(y_0) \right),$$

for $y \in \Omega_\delta = \{x \in \Omega : \rho(x) = \text{dist}(x, y_0) < \delta\}$. It follows from (5.36) that $u_\nu \leq \mu$ on $\partial\Omega \cap \partial\Omega_\delta$ while (5.32) and (5.35) imply $u_\nu(y_0) = \mu(y_0)$. Now we may proceed as it was done for the mixed normal-tangential derivatives to get the estimate $\nabla_{\nu\nu}u(y_0) \leq C$, for a uniform constant C . In fact, at the definition of the function w in (5.2) we can choose the vector field ξ as being an extension of ν and change the function μ there by the function μ defined on (5.37). Defining \tilde{w} in the same way as in (5.8), the inequality (5.9) remains true, hence the function h defined at equation (5.13) still satisfies $L[h] \leq 0$ in Ω_δ and $h \geq 0$ on $\partial\Omega_\delta \cap \Omega$, for appropriate constants a_0, b_0, c_0 and $\delta > 0$ sufficiently small. To get the inequality $h \geq 0$ on $\partial\Omega_\delta \cap \partial\Omega$ we must use that $u_\nu \leq \mu$ on $\partial\Omega \cap \partial\Omega_\delta$. Then, like it was done for the mixed normal-tangential derivatives case we get

$$(5.29) \quad u_{\nu;\nu}(y_0) \leq C.$$

Therefore $\kappa[u](y_0)$ is contained in an *a priori* bounded subset of Γ . Since

$$F[u] = f(\kappa[u]) = \Psi \geq \Psi_0 = \inf \Psi > 0$$

it follows from (1.8) that

$$\text{dist}(\kappa[u](y_0), \partial\Gamma) \geq c_0 > 0$$

for a uniform constant $c_0 > 0$. Thus $d(y_0) \geq c_0$, for a uniform constant $c_0 > 0$. \square

We are now in position to prove (5.15). We assume that $u_{\nu;\nu} \geq N_0$, where N_0 is the uniform constant defined above (otherwise we are done). By our choice of N_0 we have $\tilde{\kappa}[u] \in \Gamma'$ on $\partial\Omega$, where $\tilde{\kappa}$ are the eigenvalues of the tensor T_u defined in (5.17). Fixed $y \in \partial\Omega$, we choose Fermi coordinates centered at y as it was done in (5.18) to conclude that $\tilde{\kappa}[u](y) = (u_{1;1}, \dots, u_{n-1;n-1})$ are the eigenvalues of T_u and the principal curvatures $\kappa[u](y) = (\kappa_1, \dots, \kappa_n)$ of Σ , at $(y, u(y))$, behave as is described in (5.19) and (5.20). Therefore, since $\frac{1}{W}\tilde{\kappa}[u] \in \Gamma'$, there exists a uniform constant N_1 such that if $u_{\nu;\nu}(y) \geq N_1$ then the distance of $\kappa'[u] = \kappa'(a_i^j[u])(y)$ to $\partial\Gamma'$ is greater than $c_0/2$, where c_0 is the constant at Lemma 9. Thus

$$d(\kappa'[u](y)) \geq \frac{c_0}{2},$$

for $y \in \Lambda = \{y \in \Omega : u_{\nu;\nu}(y) \geq N_1\}$. On the other hand, it follows from (1.4) that there exists a uniform constant $\delta_0 > 0$ such that

$$(5.30) \quad \lim_{t \rightarrow \infty} f(\kappa'[u](y), t) \geq \Psi(x, u) + \delta_0$$

uniformly for $y \in \Lambda$, then we have a uniform upper bound $\kappa_n[u](y) \leq C$ for $y \in \Lambda$. This yields a uniform upper bound $u_{\nu;\nu}(y) \leq C$ for $y \in \Lambda$ and establishes (5.15).

Lemma 9. *Let $N_0 > 0$ be the constant defined in ?? above and suppose $u_{\nu;\nu} \geq N_0$. Then there exists a uniform constant $c_0 > 0$ such that*

$$d(y) = d(\tilde{\kappa}[u](y)) \geq c_0 \quad \text{on} \quad \partial\Omega.$$

Proof. Consider a point $y_0 \in \partial\Omega$ where the function $d(y)$ attains its minimum in $\partial\Omega$. It suffices to prove that $d(y_0) \geq c_0 > 0$. As above we fix Fermi coordinates (y^i) in M along $\partial\Omega$, centered at y_0 , such that y^n is the normal coordinate and the tangent coordinate vectors $\{\frac{\partial}{\partial y^\alpha}|_{y_0}\}_{\alpha < n}$ diagonalize T_u at y_0 with respect to the inner product given by $\tilde{\sigma} + \tilde{\nabla}\varphi \otimes \tilde{\nabla}\varphi$. We choose indices such that

$$\tilde{\kappa}_1(y_0) \leq \dots \leq \tilde{\kappa}_{n-1}(y_0).$$

From (5.16) the coordinate system (y^α) diagonalizes also the restriction of $\nabla^2 u$ to $T(\partial\Omega)$ at y_0 and

$$(5.31) \quad \tilde{\kappa}_\alpha(y_0) = u_{\alpha;\alpha}(y_0), \quad \alpha < n.$$

We extend ν to the coordinate neighborhood by taking its parallel transport along normal geodesics departing from $\partial\Omega$ and set

$$b_{\alpha\beta} = \Pi \left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \right) = \left\langle \nabla_{\frac{\partial}{\partial y^\alpha}} \frac{\partial}{\partial y^\beta}, \nu \right\rangle.$$

Using Lemma 6.1 of [3], we find a vector $\gamma' = (\gamma_1, \dots, \gamma_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$\gamma_1 \geq \dots \geq \gamma_{n-1} \geq 0, \quad \sum_{\alpha < n} \gamma_\alpha = 1$$

and

$$(5.32) \quad d(y_0) = \sum_{\alpha < n} \gamma_\alpha \tilde{\kappa}_\alpha(y_0) = \sum_{\alpha < n} \gamma_\alpha u_{\alpha;\alpha}(y_0).$$

Moreover,

$$(5.33) \quad \Gamma' \subset \{ \lambda' \in \mathbb{R}^{n-1} : \gamma' \cdot \lambda' > 0 \}.$$

Applying Lemma 6.2 of [3] with $\gamma_n = 0$ we get for all $y \in \partial\Omega$ near y_0

$$(5.34) \quad \sum_{\alpha < n} \gamma_\alpha T_{\alpha\alpha}(y) = \sum_{\alpha < n} \gamma_\alpha u_{\alpha;\alpha}(y) \geq \sum_{\alpha < n} \gamma_\alpha \tilde{\kappa}_\alpha(y) \geq d(y) \geq d(y_0),$$

where we have used (5.33) and $|\gamma| \leq 1$ in the second inequality. Differentiating covariantly the equality $u - \varphi = 0$ on $\partial\Omega$ we get

$$(5.35) \quad (u - \varphi)_{\xi;\eta} = -(u - \varphi)_\nu \Pi(\xi, \eta) \quad \text{on } \partial\Omega,$$

for any vector fields ξ and η tangent to $\partial\Omega$. Then, for $y \in \partial\Omega$ near y_0 we have

$$u_\nu(y) \sum_{\alpha < n} \gamma_\alpha b_{\alpha\alpha}(y) = \sum_{\alpha < n} \gamma_\alpha (\varphi - u)_{\alpha;\alpha}(y).$$

Then

$$(5.36) \quad \begin{aligned} u_\nu(y) \sum_{\alpha < n} \gamma_\alpha b_{\alpha\alpha}(y) &= \sum_{\alpha < n} \gamma_\alpha \varphi_{\alpha;\alpha}(y) - \sum_{\alpha < n} \gamma_\alpha u_{\alpha;\alpha}(y) \\ &\leq \sum_{\alpha < n} \gamma_\alpha \varphi_{\alpha;\alpha}(y) - d(y_0), \end{aligned}$$

where we used (5.34) in the last inequality. Since \underline{u} is locally strictly convex in a neighborhood of $\partial\Omega$ it follows that $\kappa'(\underline{u}_{\alpha;\beta}(y_0))$ belongs to Γ' (since $\Gamma^+ \subset \Gamma$). We point out that $\kappa'(\underline{u}_{\alpha;\beta})$ denotes the eigenvalues of $\nabla^2 \underline{u}$. We may assume

$$d(y_0) < \frac{1}{2} d(\kappa'(\underline{u}_{\alpha;\beta}(y_0))),$$

otherwise we are done. Now we use the equality $u = \underline{u}$ on $\partial\Omega$ to get

$$(u - \underline{u})_\nu \sum_{\alpha < n} \gamma_\alpha b_{\alpha\alpha} = \sum_{\alpha < n} \gamma_\alpha (\underline{u} - u)_{\alpha;\alpha},$$

on $\partial\Omega$. Therefore we conclude from (5.35), (5.33) and Lemma 6.2 of [3] that

$$\begin{aligned} (u - \underline{u})_\nu(y_0) \sum_{\alpha < n} \gamma_\alpha b_{\alpha\alpha}(y_0) &= \sum_{\alpha < n} \gamma_\alpha \underline{u}_{\alpha;\alpha}(y_0) - \sum_{\alpha < n} \gamma_\alpha u_{\alpha;\alpha}(y_0) \\ &\geq d(\kappa'(\underline{u}_{\alpha;\beta}(y_0))) - d(y_0) > \frac{1}{2}d(\underline{u}_{\alpha;\beta}(y_0)) > 0. \end{aligned}$$

Since $(u - \underline{u})_\nu \geq 0$ on $\partial\Omega$ there exist uniform positive constants $c, \delta > 0$ such that

$$\sum_{\alpha < n} \gamma_\alpha b_{\alpha\alpha}(y) \geq c > 0,$$

for every $y \in \Omega$ satisfying $\text{dist}(y, y_0) < \delta$. Hence we may define the function

$$(5.37) \quad \mu(y) = \frac{1}{\sum_{\alpha < n} \mu_\alpha b_{\alpha\alpha}(y)} \left(\sum_{\alpha < n} \mu_\alpha \varphi_{\alpha;\alpha}(y) - d(y_0) \right),$$

for $y \in \Omega_\delta = \{x \in \Omega : \rho(x) = \text{dist}(x, y_0) < \delta\}$. It follows from (5.36) that $u_\nu \leq \mu$ on $\partial\Omega \cap \partial\Omega_\delta$ while (5.32) and (5.35) imply $u_\nu(y_0) = \mu(y_0)$. Now we may proceed as it was done earlier for the mixed normal-tangential derivatives to get the estimate $\nabla_{\nu\nu} u(y_0) \leq C$, for a uniform constant C . In fact, at the definition of the function w in (5.2) we can choose the vector field ξ as being an extension of ν and change the function μ there by the function μ defined on (5.37). Defining \tilde{w} in the same way as in (5.8), the inequality (5.9) remains true, hence the function h defined at equation (5.13) still satisfies $L[h] \leq 0$ in Ω_δ and $h \geq 0$ on $\partial\Omega_\delta \cap \Omega$, for appropriate constants a_0, b_0, c_0 and $\delta > 0$ sufficiently small. To get the inequality $h \geq 0$ on $\partial\Omega_\delta \cap \partial\Omega$ we must use that $u_\nu \leq \mu$ on $\partial\Omega \cap \partial\Omega_\delta$. Then, similarly to what we had done for the mixed normal-tangential derivatives case we get

$$(5.38) \quad u_{\nu;\nu}(y_0) \leq C.$$

Therefore $\kappa[u](y_0)$ is contained in an *a priori* bounded subset of Γ . Since

$$F[u] = f(\kappa[u]) = \Psi \geq \Psi_0 = \inf \Psi > 0$$

it follows from (1.8) that

$$\text{dist}(\kappa[u](y_0), \partial\Gamma) \geq c_0 > 0$$

for a uniform constant $c_0 > 0$. Thus $d(y_0) \geq c_0$, for a uniform constant $c_0 > 0$. \square

We are now in position to prove (5.15). We assume that $u_{\nu;\nu} \geq N_0$, where N_0 is the uniform constant defined above (otherwise we are done). By our choice of N_0 we have $\tilde{\kappa}[u] \in \Gamma'$ on $\partial\Omega$, where $\tilde{\kappa}$ are the eigenvalues of the tensor T_u defined in (5.17). Fixed $y \in \partial\Omega$, we choose Fermi coordinates centered at y as it was done in (5.18) to conclude that $\tilde{\kappa}[u](y) = (u_{1;1}, \dots, u_{n-1;n-1})$ are the eigenvalues of T_u and the principal curvatures $\kappa[u](y) = (\kappa_1, \dots, \kappa_n)$ of Σ , at $(y, u(y))$, behave as is described in (5.19) and (5.20). Therefore, since $\frac{1}{W}\tilde{\kappa}[u] \in \Gamma'$, there exists a uniform constant N_1 such that if $u_{\nu;\nu}(y) \geq N_1$ then the distance of $\kappa'[u] = \kappa'(a_i^j[u])(y)$ to $\partial\Gamma'$ is greater than $c_0/2$, where c_0 is the constant at Lemma 9. Thus

$$d(\kappa'[u](y)) \geq \frac{c_0}{2},$$

for $y \in \Lambda = \{y \in \Omega : u_{\nu;\nu}(y) \geq N_1\}$. On the other hand, it follows from (1.4) that there exists a uniform constant $\delta_0 > 0$ such that

$$(5.39) \quad \lim_{t \rightarrow \infty} f(\kappa'[u](y), t) \geq \Psi(x, u) + \delta_0$$

uniformly for $y \in \Lambda$, then we have a uniform upper bound $\kappa_n[u](y) \leq C$ for $y \in \Lambda$. This yields a uniform upper bound $\nabla_{\nu\nu}u(y) \leq C$ for $y \in \Lambda$ and establishes (5.15).

6. GLOBAL BOUNDS FOR THE SECOND DERIVATIVES

This section is devoted to the proof of the global Hessian estimate. We will show that the terms of the second fundamental form b of the graph of u are bounded by above. Combined with (2.16) (see Section 2), this provides us with uniform bounds for b . Since we already have the C^1 estimate, then this information allows us to obtain the global second derivative estimate.

Proposition 10. *Suppose that conditions (1.4)-(1.10) hold and that there exists a locally strictly convex function $\chi \in C^2(\overline{\Omega})$. Let $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be an admissible solution of (1.3). Then*

$$(6.1) \quad |\nabla^2 u| \leq C \quad \text{in } \overline{\Omega},$$

where C depends on $|u|_1, \max_{\partial\Omega} |\nabla^2 u|, |\underline{u}|_2$ and other known data.

Proof. First we extend the locally strictly convex function $\chi \in C^2(\overline{\Omega})$ to $\overline{\Omega} \times \mathbb{R}$ by setting

$$\chi(x, t) = \chi(x) + t^2.$$

This extension is also locally strictly convex and we use the symbol χ also to represent it. We then define the following function on the unit tangent bundle of Σ ,

$$\tilde{\zeta}(y, \xi) = b(\xi, \xi) \exp(\phi(\tau(y)) + \beta\chi(y)),$$

where $y \in \Sigma$, ξ is a unit tangent vector to Σ at y , the function τ is the support function defined on Σ by $\tau = \langle N, \partial_t \rangle$, $\beta > 0$ is a constant to be chosen later and ϕ is a real function defined as follows. The function τ is bounded by constants depending on the bound for $|\nabla u|$. Hence, it is possible to choose $a > 0$ so that $\tau \geq 2a$. Thus, we define

$$\phi(\tau) = -\ln(\tau - a).$$

Differentiating with respect to τ ,

$$(6.2) \quad \ddot{\phi} - (1 + \epsilon)\dot{\phi}^2 = \frac{1}{(\tau - a)^2} - \frac{1 + \epsilon}{(\tau - a)^2} = -\frac{\epsilon}{(\tau - a)^2} < 0,$$

for any positive constant $\epsilon > 0$. Notice that, by the choice of a , given an arbitrary positive constant C , we have

$$(6.3) \quad -(1 + \dot{\phi}\tau) + C(\ddot{\phi} - (1 + \epsilon)\dot{\phi}^2) = -1 + \frac{\tau}{\tau - a} - \frac{c_1\epsilon}{(\tau - a)^2} \geq \frac{a^2}{2(\tau - a)^2} \geq \hat{C},$$

for some positive constant \hat{C} depending on the bound for $|\nabla u|$.

If the maximum of $\tilde{\zeta}$ is achieved on $\partial\Sigma$, we can estimate it in terms of uniform constants (see the last section) and we are done. Thus, suppose the maximum of $\tilde{\zeta}$ is attained at a point $y_0 = (x_0, u(x_0)) \in \Sigma$, with $x_0 \in \Omega$, and along the direction ξ_0 tangent to Σ at $y_0 = (x_0, u(x_0))$. We fix a normal coordinate system (y^i) of Σ centered at y_0 such that

$$\frac{\partial}{\partial y^1} \Big|_{y_0} = \xi_0.$$

Notice that ξ_0 is a principal direction of Σ at y_0 , thus $a_{1i}(y_0) = 0$, for any $i > 1$. Consider the local function $a_{11} = b(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1})$. Then the function

$$(6.4) \quad \zeta = a_{11} \exp(\phi(\tau) + \beta\chi)$$

attains maximum at $y_0 = (x_0, u(x_0))$. Hence, it holds at y_0

$$(6.5) \quad 0 = (\ln \zeta)_i = \frac{a_{11;i}}{a_{11}} + \dot{\phi}\tau_i + \beta\chi_i$$

and the Hessian matrix with components

$$(\ln \zeta)_{i;j} = \frac{a_{11;ij}}{a_{11}} - \frac{a_{11;i}a_{11;j}}{a_{11}^2} + \dot{\phi}\tau_{i;j} + \ddot{\phi}\tau_i\tau_j + \beta\chi_{i;j}$$

is negative-definite. Thus

$$(6.6) \quad \begin{aligned} G^{ij}(\ln \zeta)_{i;j} &= \frac{1}{a_{11}} G^{ij} a_{11;ij} - \frac{1}{a_{11}^2} G^{ij} a_{11;i} a_{11;j} + \dot{\phi} G^{ij} \tau_{i;j} \\ &\quad + \ddot{\phi} G^{ij} \tau_i \tau_j + \beta G^{ij} \chi_{i;j} \leq 0. \end{aligned}$$

We may rotate the coordinates (y^2, \dots, y^n) in such a way that the new coordinates diagonalize $\{a_{ij}(y_0)\}$. By Lemma 3 $\{G^{ij}\}$ is also diagonal with $G^{ii} = \frac{1}{W} f_i$. We denote $\kappa_i = a_{ii}(y_0)$ and choose indices in such a way that

$$\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_n.$$

Moreover, we assume without loss of generality that $\kappa_1 > 1$ at y_0 . Thus, according to Lemma 3, we have

$$f_1 \leq f_2 \leq \dots \leq f_n.$$

From (6.6),

$$(6.7) \quad \sum_i \left(\frac{1}{\kappa_1} f_i a_{11;ii} - \frac{1}{\kappa_1^2} f_i |a_{11;i}|^2 + \dot{\phi} f_i \tau_{i;i} + \ddot{\phi} f_i |\tau_i|^2 + \beta f_i \chi_{i;i} \right) \leq 0.$$

Now, we differentiate covariantly with respect to the metric (g_{ij}) in Σ the equation (2.11) in the direction of $\frac{\partial}{\partial y^1}|_{y_0}$ obtaining $F^{ij} a_{ij;1} = \Psi_1$ and differentiating again

$$(6.8) \quad F^{ij} a_{ij;11} + F^{ij,kl} a_{ij;1} a_{kl;1} = \Psi_{1;1}.$$

From the Simons formula (2.9) we have

$$(6.9) \quad \begin{aligned} F^{ij} a_{ij;11} &= F^{ii} a_{ii;11} = \sum_i (f_i a_{11;ii} + \kappa_1 f_i \kappa_i^2 - \kappa_1^2 f_i \kappa_i \\ &\quad + \kappa_1 f_i \bar{R}_{i0i0} - \bar{R}_{1010} f_i \kappa_i + f_i \bar{R}_{1i0;1} - f_i \bar{R}_{1i0;i}). \end{aligned}$$

It follows from $c_0 \leq \sum_i f_i \lambda_i \leq f = \Psi$ that

$$\begin{aligned} F^{ij} a_{ij;11} &\leq -\kappa_1^2 c_0 + |\bar{R}_{1010}| \Psi \\ &\quad + \sum_i (f_i a_{11;ii} + \kappa_1 f_i \kappa_i^2 + \kappa_1 f_i \bar{R}_{i0i0} + f_i \bar{R}_{i0i0;1} - f_i \bar{R}_{1010;i}). \end{aligned}$$

Combining this expression and (6.8) we obtain

$$\begin{aligned} \sum_i f_i a_{11;ii} &\geq \Psi_{1;1} - F^{ij,kl} a_{ij;1} a_{kl;1} + \kappa_1^2 c_0 - |\bar{R}_{1010}| \psi \\ &\quad - \sum_i (\lambda_1 f_i \lambda_i^2 - \lambda_1 f_i \bar{R}_{i0i0} - f_i \bar{R}_{i0i0;1} + f_i \bar{R}_{1010;i}). \end{aligned}$$

Replacing this equation into (6.7) we get

$$\begin{aligned} & \frac{1}{\kappa_1} (\Psi_{1;1} - F^{ij,kl} a_{ij;1} a_{kl;1} + \kappa_1^2 c_0 - |\bar{R}_{1010}| \Psi) \\ & - \frac{1}{\kappa_1} \sum_i (\kappa_1 f_i \kappa_i^2 - \kappa_1 f_i \bar{R}_{i0i0} - f_i \bar{R}_{i0i0;1} + f_i \bar{R}_{1010;i}) \\ & + \sum_i \left(\dot{\phi} f_i \tau_{i;i} - \frac{1}{\kappa_1^2} f_i |a_{11;i}|^2 + \ddot{\phi} f_i |\tau_i|^2 + \beta f_i \chi_{i;i} \right) \leq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{\Psi_{1;1}}{\kappa_1} + \frac{1}{\kappa_1} (c_0 \kappa_1^2 - \Psi |\bar{R}_{1010}|) - \frac{1}{\kappa_1} F^{ij,kl} a_{ij;1} a_{kl;1} - \sum_i f_i \kappa_i^2 \\ & - \sum_i f_i \bar{R}_{i0i0} + \sum_i \left(\dot{\phi} f_i \tau_{i;i} - \frac{1}{\kappa_1^2} f_i |a_{11;i}|^2 + \ddot{\phi} f_i |\tau_i|^2 + \beta f_i \chi_{i;i} \right) \\ & - \frac{1}{\kappa_1} \sum_i f_i (\bar{R}_{i0i0;1} - \bar{R}_{1010;i}) \leq 0. \end{aligned}$$

It is well known that

$$\begin{aligned} \tau_i &= -a_i^k \eta_k \\ \tau_{i;j} &= -\eta^k a_{ii;k} - \eta^k \bar{R}_{kij0} - \tau a_i^k a_{kj}, \end{aligned}$$

where η^k are the components of the vector ∂_t^T , which is the projection of ∂_t onto $T\Sigma$. Hence,

$$\dot{\phi} \sum_i f_i \tau_{i;i} = -\dot{\phi} \left(\sum_i \eta^k f_i a_{ii;k} + \sum_i \eta^k \bar{R}_{kii0} f_i \right) - \dot{\phi} \tau \sum_i f_i \kappa_i^2.$$

From $\sum_i f_i a_{ii;k} = \Psi_k$,

$$\dot{\phi} \sum_i f_i \tau_{i;i} = -\dot{\phi} \left(\eta^k \Psi_k + \sum_i \eta^k \bar{R}_{kii0} f_i \right) - \dot{\phi} \tau \sum_i f_i \kappa_i^2.$$

We denote by $T = \sum_i f_i$. By estimating the ambient curvature terms,

$$\sum_i \eta^k \bar{R}_{kii0} f_i \leq CT.$$

Then,

$$-\dot{\phi} \left(\eta^k \Psi_k + \sum_i \eta^k \bar{R}_{kii0} f_i \right) \geq -|\dot{\phi}|(C + CT).$$

Therefore

$$\dot{\phi} \sum_i f_i \tau_{i;i} \geq -|\dot{\phi}|(C + CT) - \dot{\phi} \tau \sum_i f_i \kappa_i^2.$$

Now, we suppose without loss of generality that

$$\kappa_1 \geq \frac{1}{C} \sum_i |R_{i0i0;1} - R_{1010;i}|, \quad -\frac{1}{\kappa_1} \Psi |\bar{R}_{1010}| \geq -C, \quad \text{and} \quad \frac{\Psi_{1;1}}{\kappa_1} \geq -C$$

for a positive constant $C > 0$. Since

$$\Psi_{1;1} = \Psi_{t;t}(u_1)^2 + \Psi_t u_{1;1} + \Psi_{1;1},$$

the above assumptions is allowed. Finally,

$$-\sum_i f_i \bar{R}_{i0i0} \geq -T \max_i |\bar{R}_{i0i0}| \geq -CT.$$

We conclude from these inequalities that

$$(6.10) \quad \begin{aligned} & -C - CT + c_0 \kappa_1 - \frac{1}{\kappa_1} F^{ij,kl} a_{ij;1} a_{kl;1} - \sum_i f_i \kappa_i^2 - \frac{1}{\kappa_1^2} \sum_i f_i |a_{11;i}|^2 \\ & - |\dot{\phi}|(C + CT) - \dot{\phi} \tau \sum_i f_i \kappa_i^2 + \ddot{\phi} \sum_i f_i |\tau_i|^2 + \beta \sum_i f_i \chi_{i;i} \leq 0. \end{aligned}$$

Now, to proceed further with our analysis we consider two cases.

Case I: In this case we suppose that $\kappa_n \leq -\theta \kappa_1$ for some positive constant θ to be chosen later.

Using (6.5) and the Cauchy inequality we get

$$(6.11) \quad \frac{1}{\kappa_1^2} f_i |a_{11;i}|^2 = f_i |\dot{\phi} \tau_i + \beta \chi_i|^2 \leq (1 + \frac{1}{\epsilon}) \beta^2 f_i |\chi_i|^2 + (1 + \epsilon) \dot{\phi}^2 f_i |\tau_i|^2,$$

for any $\epsilon > 0$ and any $1 \leq i \leq n$. Now we replace the sum of the terms in (6.11) in the inequality (6.10) to obtain

$$\begin{aligned} & c_0 \kappa_1 - C(1 + |\dot{\phi}|) - CT(1 + |\dot{\phi}|) - \frac{1}{\kappa_1} F^{ij,kl} a_{ij;1} a_{kl;1} - (1 + \dot{\phi} \tau) \sum_i f_i \kappa_i^2 \\ & - (1 + \frac{1}{\epsilon}) \beta^2 \sum_i f_i |\chi_i|^2 + (\ddot{\phi} - (1 + \epsilon) \dot{\phi}^2) \sum_i f_i |\tau_i|^2 + \beta \sum_i f_i \chi_{i;i} \leq 0. \end{aligned}$$

Since $\{a_{ij}\}$ is diagonal at y_0 ,

$$\sum_i f_i |\tau_i|^2 = \sum_i f_i \lambda_i^2 |\eta_i|^2 \leq C \sum_i f_i \kappa_i^2,$$

so, it follows from (6.2) that

$$(\ddot{\phi} - (1 + \epsilon) \dot{\phi}^2) \sum_i f_i |\tau_i|^2 \geq (\ddot{\phi} - (1 + \epsilon) \dot{\phi}^2) C \sum_i f_i \kappa_i^2.$$

Since $|D\chi|$ is a known data we have $\sum_i f_i |\chi_i|^2 \leq CT$. Hence,

$$(6.12) \quad \begin{aligned} & c_0 \kappa_1 - C(1 + |\dot{\phi}|) - \frac{1}{\kappa_1} F^{ij,kl} a_{ij;1} a_{kl;1} - (1 + |\dot{\phi}| + (1 + \frac{1}{\epsilon}) \beta^2) CT \\ & + \left(- (1 + \dot{\phi} \tau) + C(\ddot{\phi} - (1 + \epsilon) \dot{\phi}^2) \right) \sum_i f_i \kappa_i^2 + \beta \sum_i f_i \chi_{i;i} \leq 0. \end{aligned}$$

Using the concavity of F and the convexity of χ we may discard the third and the last terms in the left-hand side of (6.12) since they are nonnegative, obtaining

$$-C_1(\beta) - C_2(\beta)T + c_0 \kappa_1 + \hat{C} \sum_i f_i \kappa_i^2 \leq 0,$$

where C_1 depends linearly on β and C_2 depends quadratically on β . Since $f_n \geq \frac{1}{n}T$, we have

$$\sum_i f_i \kappa_i^2 \geq f_n \kappa_n^2 \geq \frac{1}{n} \theta^2 T \kappa_1^2.$$

Thus it follows that

$$(6.13) \quad -C_1 - C_2T + c_0\kappa_1 + \hat{C}\frac{1}{n}\theta^2T\kappa_1^2 \leq 0.$$

This inequality shows that κ_1 has a uniform upper bound. In fact, notice that the coefficients of the terms in T in (6.13) are

$$\hat{C}\frac{1}{n}\theta^2\kappa_1^2 - C_2.$$

Then, if $\kappa_1 \geq \bar{C}$ for a (suitable) uniform constant \bar{C} , we have

$$\hat{C}\frac{1}{n}\theta^2\kappa_1^2 - C_2 \geq 0.$$

In this case, since $T = \sum_i f_i \geq 0$, we may discard the terms in T in (6.13) to obtain $-C_1 + c_0\kappa_1 \leq 0$, i.e.,

$$\kappa_1 \leq \frac{C_1}{c_0}.$$

Case II: In this case we assume that $\kappa_n \geq -\theta\kappa_1$. Hence, $\kappa_i \geq -\theta\kappa_1$. We then group the indices $\{1, \dots, n\}$ in two sets

$$\begin{aligned} I_1 &= \{j; f_j \leq 4f_1\}, \\ I_2 &= \{j; f_j > 4f_1\}. \end{aligned}$$

Using (6.11), we have for $i \in I_1$

$$\begin{aligned} \frac{1}{\kappa_1^2} f_i |a_{11;i}|^2 &\leq (1 + \epsilon) \dot{\phi}^2 f_i |\tau_i|^2 + (1 + \frac{1}{\epsilon}) (\beta)^2 f_i |\chi_i|^2 \\ &\leq (1 + \epsilon) \dot{\phi}^2 f_i |\tau_i|^2 + C(1 + \frac{1}{\epsilon}) (\beta)^2 f_1. \end{aligned}$$

Therefore, it follows from (6.11) that

$$\begin{aligned} &-C - CT + c_0\kappa_1 - \frac{1}{\lambda_1} F^{ij,kl} a_{ij;1} a_{kl;1} - (1 + \dot{\phi}\tau) \sum_i f_i \kappa_i^2 - \frac{1}{\kappa_1^2} \sum_{j \in I_2} f_j |a_{11;j}|^2 \\ &- |\dot{\phi}|(C + CT) + (\ddot{\phi} - (1 + \epsilon)\dot{\phi}^2) \sum_i f_i |\tau_i|^2 - C(1 + \frac{1}{\epsilon}) \beta^2 f_1 + \beta \sum_i f_i \chi_{i;i} \leq 0. \end{aligned}$$

Notice that we had summed up to the inequality the non-positive terms

$$-(1 + \epsilon) |\dot{\phi}|^2 \sum_{i \in I_2} f_i |\tau_i|^2.$$

Using that $|\tau_i| = |\kappa_i \eta_i| \leq C\kappa_i$ we may conclude as above that

$$(6.14) \quad - (1 + \dot{\phi}\tau) \sum_i f_i \kappa_i^2 + (\ddot{\phi} - (1 + \epsilon)\dot{\phi}^2) \sum_i f_i |\tau_i|^2 \geq \hat{C} \sum_i f_i \kappa_i^2$$

for some positive constant $\hat{C} > 0$. Thus

$$\begin{aligned} &-C - CT + c_0\kappa_1 - \frac{1}{\kappa_1} F^{ij,kl} a_{ij;1} a_{kl;1} + \hat{C} \sum_i f_i \kappa_i^2 \\ (6.15) \quad &- \frac{1}{\kappa_1^2} \sum_{j \in I_2} f_j |a_{11;j}|^2 - |\dot{\phi}|(C + CT) - C(1 + \frac{1}{\epsilon}) \beta^2 f_1 + \beta \sum_i f_i \chi_{i;i} \leq 0. \end{aligned}$$

Using the Codazzi equation $a_{1j;1} = a_{11;j} + \bar{R}_{01j1}$ and Lemma 3 we get

$$\begin{aligned} -\frac{1}{\kappa_1} F^{ij,kl} a_{ij;1} a_{kl;1} &= -\frac{1}{\kappa_1} \sum_{k,l} f_{kl} a_{kk;1} a_{ll;1} - \frac{1}{\kappa_1} \sum_{k \neq l} \frac{f_k - f_l}{\kappa_k - \kappa_l} \eta_{kl}^2 \\ &\geq -\frac{2}{\kappa_1} \sum_{j \in I_2} \frac{f_1 - f_j}{\kappa_1 - \kappa_j} (a_{1j;1})^2 \\ &= -\frac{2}{\kappa_1} \sum_{j \in I_2} \frac{f_1 - f_j}{\kappa_1 - \kappa_j} (a_{11;j} + \bar{R}_{01j1})^2, \end{aligned}$$

since $1 \notin I_2$ and $\frac{f_k - f_l}{\kappa_k - \kappa_l} \leq 0$. We claim that for all $j \in I_2$ it holds the inequality

$$(6.16) \quad -\frac{2}{\kappa_1} \frac{f_1 - f_j}{\kappa_1 - \kappa_j} \geq \frac{f_j}{\kappa_1^2}.$$

This is equivalent to

$$2f_1\kappa_1 \leq f_j\kappa_1 + f_j\kappa_j.$$

It is clear that $j \in I_2$ implies $f_j > 4f_1$. If $\kappa_j \geq 0$, this is obvious. If $\kappa_j < 0$, then $-\theta\kappa_1 \leq \kappa_j < 0$, and then

$$f_j\kappa_1 + f_j\kappa_j \geq (1 - \theta)f_j\kappa_1 \geq 4(1 - \theta)f_1\kappa_1 \geq 2f_1\kappa_1,$$

if we choose $\theta = 1/2$. Hence, with this choice, we can use (6.16) to obtain

$$\begin{aligned} -\frac{1}{\kappa_1} F^{ij,kl} a_{ij;1} a_{kl;1} &\geq \sum_{j \in I_2} \frac{f_j}{\kappa_1^2} (a_{11;j} + \bar{R}_{01j1})^2 \\ &= \sum_{j \in I_2} \frac{f_j}{\kappa_1^2} (a_{11;j})^2 + 2 \sum_{j \in I_2} \frac{f_j}{\kappa_1^2} a_{11;j} \bar{R}_{01j1} + \sum_{j \in I_2} \frac{f_j}{\kappa_1^2} (\bar{R}_{01j1})^2. \end{aligned}$$

Using this inequality in (6.15) and estimating the curvature term $(R_{01j1})^2$ we obtain

$$\begin{aligned} &-C - CT + c_0\kappa_1 + \sum_{j \in I_2} \frac{f_j}{\kappa_1^2} (a_{11;j})^2 + 2 \sum_{j \in I_2} \frac{f_j}{\kappa_1^2} a_{11;j} \bar{R}_{01j1} + \hat{C} \sum_i f_i \kappa_i^2 \\ &- \frac{1}{\kappa_1^2} \sum_{j \in I_2} f_j |a_{11;j}|^2 - |\dot{\phi}| (C + CT) - C(1 + \frac{1}{\epsilon}) \beta^2 f_1 + \beta \sum_i f_i \chi_{i;i} \leq 0. \end{aligned}$$

From (6.5),

$$\begin{aligned} &-C - CT + c_0\kappa_1 - 2 \sum_{j \in I_2} \frac{f_j}{\kappa_1} (\dot{\phi}\tau_j + \beta\chi_j) \bar{R}_{01j1} + \hat{C} \sum_i f_i \kappa_i^2 \\ &- |\dot{\phi}| (C + CT) - C(1 + \frac{1}{\epsilon}) \beta^2 f_1 + \beta \sum_i f_i \chi_{i;i} \leq 0. \end{aligned}$$

Since $\dot{\phi} < 0$, $\kappa_j \leq \kappa_1$ and $-\kappa_j \leq \theta\kappa_1 < \kappa_1$ we have

$$2 \frac{f_j}{\kappa_1} (-\dot{\phi}\tau_j) \bar{R}_{01j1} = 2 \frac{f_j}{\kappa_1} \dot{\phi} \kappa_j \eta_j \bar{R}_{01j1} \geq 2 \frac{f_j}{\kappa_1} \dot{\phi} |\kappa_j| |\eta_j \bar{R}_{01j1}| \geq 2 f_j \dot{\phi} |\eta_j \bar{R}_{01j1}|.$$

We also suppose, without loss of generality, that

$$\kappa_1 \geq \frac{3|\chi_j \bar{R}_{01j1}|}{\gamma_0}$$

for all $j \in I_2$, where γ_0 is a positive constant that satisfies

$$\chi_{i;i} \geq \gamma_0 > 0, \quad \forall 1 \leq i \leq n.$$

Note that this assumption is equivalent to

$$\frac{\gamma_0}{3} \geq \frac{|\chi_j \bar{R}_{01j1}|}{\kappa_1},$$

which implies

$$-2 \sum_{j \in I_2} \frac{f_j}{\kappa_1} \beta \chi_j \bar{R}_{01j1} \geq -2 \sum_{j \in I_2} \frac{f_j}{\kappa_1} \beta |\chi_j \bar{R}_{01j1}| \geq -2 \sum_{j \in I_2} \frac{\beta f_j \gamma_0}{3} \geq -2 \frac{\beta \gamma_0}{3} T.$$

These inequalities imply that

$$\begin{aligned} & -C - CT + c_0 \kappa_1 + 2 \sum_{j \in I_2} f_j \dot{\phi} |\eta_j \bar{R}_{01j1}| - 2 \frac{\beta \gamma_0}{3} T \\ & + \hat{C} \sum_i f_i \kappa_i^2 - |\dot{\phi}| (C + CT) - C \left(1 + \frac{1}{\epsilon}\right) \beta^2 f_1 + \beta \sum_i f_i \chi_{i;i} \leq 0. \end{aligned}$$

Since $\sum_{j \in I_2} f_j \leq T$, $|\eta_j \bar{R}_{j1}| \leq C$ and $\dot{\phi} < 0$ we have

$$-C - (C + C|\dot{\phi}| + 2\beta \frac{\gamma_0}{3} - \beta \gamma_0) T - C \left(1 + \frac{1}{\epsilon}\right) \beta^2 f_1 + c_0 \kappa_1 + \hat{C} f_1 \kappa_1^2 \leq 0.$$

Choosing $\beta > 0$ sufficiently large, the term in T is positive and we may discard it, obtaining

$$(6.17) \quad -C - C_2(\beta) f_1 + c_0 \kappa_1 + \hat{C} f_1 \kappa_1^2 \leq 0,$$

where C_2 depends quadratically on β . Reasoning as above, we conclude that this inequality gives an upper bound for κ_1 . \square

7. PROOF OF THE EXISTENCE –THE CONTINUITY METHOD

In this section we complete the proof of Theorems 1 and 2 by the continuity method with the aid of the *a priori* estimates established previously. We apply the continuity method to the family of problems

$$(7.1) \quad \begin{cases} F[u] = t\Psi + (1-t)F[\chi_0] & \text{in } \Omega \\ u = t\varphi + (1-t)\chi_0 & \text{on } \partial\Omega, \end{cases}$$

for $0 \leq t \leq 1$, where χ_0 is a multiple of the locally strictly convex function $\chi \in C^2(\bar{\Omega})$ that satisfy

$$0 < F[\chi_0] \leq \Psi.$$

Clearly all our preceding estimates are independent of the parameter t , so that under the hypotheses of Theorem 1 and 2, we conclude an *a priori* estimate of the form

$$|u|_{2,\alpha} \leq C$$

with constant C depending on $n, \Omega, \underline{u}, \chi, \varphi$ and Ψ , and hence the unique solvability of the Dirichlet problems (7.1), for all $0 \leq t \leq 1$, then follows.

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